One Control Problem for a Set-valued Object

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Abstract— **The article presents some definitions of derivatives for set-valued mappings and their properties. A linear set-valued differential equation is considered and conditions for the existence of basic solutions are given. Subsequently, one optimal control problem is considered, when the system behavior is described by linear set-valued differential equations.**

Keywords—set-valued mapping; Hukuhara derivative; control; linear; differential equation

I. Introduction

Recently, within the framework of the theory of set-valued equations, properties of solutions of set-valued differential equations, set-valued differential inclusions, set-valued integro-differential equations, and set-valued integral equations were considered, as well as impulse set-valued systems and controlled set-valued systems (see [1-4] and references in them). Obviously, obtaining all these results was impossible without the development of the theory of set-valued analysis. In the last decades in the works of A.V. Plotnikov and N.V. Skripnik [5], M.T. Malinowski [10], H. Vu, L.S. Dong [11], and S.E. Amrahov, A. Khastan, N. Gasilov, A.G. Fatullayev [12] definitions of the derivative of a set-valued mapping were introduced, that unlike the already classical Hukuhara derivative [13], made it possible to differentiate set-valued mappings whose diameter is a decreasing function. This made it possible not only to consider new types of set-valued differential equations, but also to formulate new optimal control problems.

One of these problems - the task of speed and the method for its solution are considered in this article.

II. Preliminaries

Let R be the set of real numbers and R^n be the n-dimensional Euclidean space ($n \geq 2$). Denote by $conv(R^n)$ the set of nonempty compact and convex subsets of R^n. For two given sets $X, Y \in conv(R^n)$ and $\lambda \in R$, the Minkowski sum and scalar multiple are defined by $X + Y = \{x + y \mid x \in X, y \in Y\}$ *and* $\lambda X = \{\lambda x \mid x \in X\}$.

Consider the Hausdorff distance $h(\cdot,\cdot)$ given by $h(X,Y) = \min\{r \geq 0 \mid X \subset Y + B_r(0), Y \subset X + B_r(0)\}$, where $B_r(0) = \{x \in R^n \mid \|x\| \leq r\}$ is the closed ball with radius r centered at the origin ($\|x\|$ denotes the Euclidean norm). It is known that $(conv(R^n), h)$ is a complete metric space. However, $conv(R^n)$ is not a linear space since it does not contain inverse elements for the addition, and therefore difference is not well defined, i.e. if $A \in conv(R^n)$ and $A \neq \{a\}$, then $A + (-1)A \neq \{0\}$. As a consequence, alternative formulations for difference have been suggested. One of these alternatives is the Hukuhara difference [13]. Let $X, Y \in conv(R^n)$. A set $Z \in conv(R^n)$ such that $X = Y + Z$ is called a Hukuhara difference (H-difference) of the sets X and Y and is denoted by $X \overset{H}{-} Y$. In this case $X \overset{H}{-} X = \{0\}$ and also $(A + B) \overset{H}{-} B = A$ for any $A, B \in conv(R^n)$.

Simultaneously, M. Hukuhara introduced the concept of H-differentiability [13] for set-valued mappings by using the H-difference.

Definition 1 [13]. Let $X : [0,T] \to conv(R^n)$ and $t \in [0,T]$. We say that $X(\cdot)$ has a H-derivative $D_H X(t) \in conv(R^n)$ at $t \in (0,T)$, if for all $\Delta > 0$ that are sufficiently close to 0, the H-differences and the limits exist

$$\lim_{\Delta \to 0} \Delta^{-1}(X(t+\Delta) \overset{H}{-} X(t)) =$$

$$= \lim_{\Delta \to 0} \Delta^{-1}(X(t) \overset{H}{-} X(t-\Delta)) = D_H X(t).$$

Theorem 1 [13]. If the mapping $X:[0,T]\to conv(R^n)$ is H-differentiable on $[0,T]$, then $X(t)=X(0)+\int_0^t D_H X(s)ds,$ where the integral is understood in the sense of [13].

Corollary 1. If the set-valued mapping $X(\cdot)$ is H-differentiable on $[0,T]$, then $diam(X(\cdot))$ is a non-decreasing function on $[0,T]$.

Corollary 2. If the function $diam(X(\cdot))$ is a decreasing function on $[0,T]$, then the set-valued mapping $X(\cdot)$ is not H-differentiable on $[0,T]$.

Subsequently, other definitions of derivatives were introduced for set-valued mappings to eliminate this disadvantage. A.V. Plotnikov and N.V. Skripnik took advantage of some approaches that were used in [4,14] and introduced a new definition of a derivative, and studied its properties [5-9].

Definition 2 [5]. Let $X:[0,T]\to conv(R^n)$ and $t\in[0,T]$. We say that $X(\cdot)$ has a PS-derivative $D_{ps}X(t)\in conv(R^n)$ at $t\in(0,T)$, if for all $\Delta>0$ that are sufficiently close to 0, the H-differences and the limits exist in at least one of the following expressions:

$$\lim_{\Delta\to0}\Delta^{-1}(X(t+\Delta)\overset{H}{-}X(t))= =\lim_{\Delta\to0}\Delta^{-1}(X(t)\overset{H}{-}X(t-\Delta))=D_{ps}X(t), \tag{i}$$

$$\lim_{\Delta\to0}\Delta^{-1}(X(t)\overset{H}{-}X(t+\Delta))= =\lim_{\Delta\to0}\Delta^{-1}(X(t-\Delta)\overset{H}{-}X(t))=D_{ps}X(t), \tag{ii}$$

$$\lim_{\Delta\to0}\Delta^{-1}(X(t+\Delta)\overset{H}{-}X(t))= =\lim_{\Delta\to0}\Delta^{-1}(X(t-\Delta)\overset{H}{-}X(t))=D_{ps}X(t), \tag{iii}$$

$$\lim_{\Delta\to0}\Delta^{-1}(X(t)\overset{H}{-}X(t+\Delta))= =\lim_{\Delta\to0}\Delta^{-1}(X(t)\overset{H}{-}X(t-\Delta))=D_{ps}X(t). \tag{iv}$$

Theorem 2 [5]. If the mapping $X:[0,T]\to conv(R^n)$ is PS-differentiable on $[0,T]$, then if the function $diam(X(t))$ is a non-decreasing (decreasing) function on $[0,T]$, then

$$X(t)=X(0)+\int_0^t D_{ps}X(s)ds \quad \left(X(t)=X(0)\overset{H}{-}\int_0^t D_{ps}X(s)ds\right).$$

Later, M.T. Malinowski [10], H. Vu and L.S. Dong [11], and S .E. Amrahov, A. Khastan, N. Gasilov and A.G. Fatullayev [12] adapted the concept of the Bede-Gal derivative [15] for interval-valued mappings on set-valued mappings, and studied its properties [10-12].

Definition 3 [12]. Let $X:[0,T]\to conv(R^n)$ and $t\in[0,T]$. We say that $X(\cdot)$ has a BG-derivative $D_{bg}X(t)\in conv(R^n)$ at $t\in(0,T)$, if for all $\Delta>0$ that are sufficiently close to 0, the H-differences and the limits exist in at least one of the following expressions:

$$\lim_{\Delta\to0}\Delta^{-1}(X(t+\Delta)\overset{H}{-}X(t))= =\lim_{\Delta\to0}\Delta^{-1}(X(t)\overset{H}{-}X(t-\Delta))=D_{bg}X(t), \tag{i}$$

$$\lim_{\Delta\to0}(-\Delta)^{-1}(X(t)\overset{H}{-}X(t+\Delta))= =\lim_{\Delta\to0}(-\Delta)^{-1}(X(t-\Delta)\overset{H}{-}X(t))=D_{bg}X(t), \tag{ii}$$

$$\lim_{\Delta\to0}\Delta^{-1}(X(t+\Delta)\overset{H}{-}X(t))= =\lim_{\Delta\to0}(-\Delta)^{-1}(X(t-\Delta)\overset{H}{-}X(t))=D_{bg}X(t), \tag{iii}$$

$$\lim_{\Delta\to0}(-\Delta)^{-1}(X(t)\overset{H}{-}X(t+\Delta))= =\lim_{\Delta\to0}\Delta^{-1}(X(t)\overset{H}{-}X(t-\Delta))=D_{bg}X(t). \tag{iv}$$

Remark 1. In [10], M.T. Malinowski considered set-valued mappings that satisfy condition (ii) and called this derivative a second type Hukuhara derivative.

Theorem 3 [12]. If the mapping $X:[0,T]\to conv(R^n)$ is BG-differentiable on $[0,T]$, then if the function $diam(X(t))$ is a non-decreasing (decreasing) function on $[0,T]$, then

$$X(t)=X(0)+\int_0^t D_{bg}X(s)ds \quad \left(X(t)=X(0)\overset{H}{-}(-1)\int_0^t D_{bg}X(s)ds\right).$$

Remark 2. If the set-valued mapping $X(\cdot)$ is H-differentiable on $[0,T]$ then it is BG-differentiable on $[0,T]$ and PS-differentiable on $[0,T]$ as well as $D_H X(t)=D_{ps}X(t)=D_{bg}X(t)$.

Remark 3. There exist set-valued mappings $X(\cdot)$ such that $D_{bg}X(t)\neq D_{ps}X(t)$ for any t.

III. LINEAR SET-VALUED DIFFERENTIAL EQUATIONS

Now, we consider linear set-valued differential equations

$$DX(t)=aX(t),\quad X(0)=X_0, \tag{1}$$

where $a\in R,\ (a\neq0)$; $X:[0,T]\to conv(R^n)$ is a set-valued mapping; $DX(t)$ is one of the previously considered derivatives $(D_H X(t), D_{ps} X(t), D_{bg}(t))$ of the set-valued mapping $X(t)$; $X_0 \in conv(R^n)$.

Definition 4. A set-valued mapping $X(\cdot)$ is called a solution of (1) if it is continuously differentiable and satisfies system (1) everywhere on $[0,T]$.

As known, linear differential equation (1) with Hukuhara has a unique solution on the interval $[0,T]$. It is also obvious that function $diam(X(t))$ is a non-decreasing function on $[0,T]$.

Now, we consider linear set-valued differential equation (1) with PS-derivative and BG-derivative. By [5-12], set-valued differential equation (1) with PS(BG)-derivative has at least one solution. Moreover, one of these solutions (the one whose diameter is a non-decreasing function) coincides with the solution of the corresponding Hukuhara differential equation and always exists. The second solution that may exist is such that its diameter is a decreasing function. These solutions are called basic (first and second basic solutions). There may also be mixed solutions such that the diameter is not a monotonic function.

Proposition 1. For set-valued differential equation (1) with PS(BG)-derivative the following statements are true:

1) if H-difference $X_0 \overset{H}{-} (-1)X_0$ exists, then differential equation (1) with PS(BG)-derivative has two basic solutions;

2) if H-difference $X_0 \overset{H}{-} (-1)X_0$ does not exist, then

a) if $a > 0$, then differential equation (1) with PS-derivative has two basic solutions and differential equation (1) with BG-derivative has one basic solution;

b) if $a < 0$, then differential equation (1) with BG-derivative has two basic solutions and differential equation (1) with PS-derivative has one basic solution.

IV. LINEAR CONTROL PROBLEM

Let the object behavior be described by the following system

$$D_{ps} X = X + u, \quad X(0,u) = X_0, \tag{2}$$

where $X:[0,T] \to conv(R^n)$ is a set-valued mapping; $u \in U \in conv(R^n)$ is a control vector; $X_0 \in conv(R^n)$; $0 \in \operatorname{int} U$.

A vector function $u(\cdot)$ is called an admissible control for the system (2) on an interval $[0,T]$ if it is summable on $[0,T]$ and $u(t) \in U$ for all $t \in [0,T]$.

The set-valued mapping $X(\cdot,u)$ will be called the solution of the system (2) on the interval $[0,T]$ corresponding to the admissible control $u(\cdot)$.

We also note that any solution $X(\cdot,u)$ of the system (2) has the following property: for all $t \in [0,T]$, there are always such a number $b(t) > 0$ and the vector $c(t) \in R^n$ that $X(t,u) = b(t)X_0 + c(t)$, that is, the section of any solution of the system (2) preserves the shape of the initial set.

Let $X_K \in conv(R^n)$. Suppose that X_0 and X_K are homothetic figures with coefficient $k > 0$.

Consider the following problem of speed: it is required to move the object $X(\cdot,u)$ according to the system (2) from the initial set X_0 to the final set X_K for the minimum time $T > 0$, so that $X(T,u) = X_K$.

In paper [16], a similar problem was considered for the case when the behavior of the system was described by a controlled differential equation with the Hukuhara derivative and at the final moment of time the condition $X(T,u) \cap X_K \neq \varnothing$ should be satisfied.

Let $X_0 = B_1(0)$. Consequently, $X_K = B_b(c)$.

At the beginning, we consider the classical control problems:

1) it is required to move the object $x_1(\cdot,u)$ according to system $\dot{x}_1 = x_1 + u$ from the starting point $x_0 = 0$ to the end point $x_K = c$ for the minimum time $T > 0$ and find the optimal control $u_*^1(\cdot)$ and the minimum time T_*^1.

2) it is required to move the object $x_2(\cdot,u)$ according to system $\dot{x}_2 = -x_2 - u$ from the starting point $x_0 = 0$ to the end point $x_K = c$ for the minimum time $T > 0$ and find the optimal control $u_*^2(\cdot)$ and the minimum time T_*^2.

Let us substitute the controls $u_*^1(\cdot)$ and $u_*^2(\cdot)$ into the system (2) and choose the first basic solution $X_1(\cdot,u_*^1)$ and the second basic solution $X_2(\cdot,u_*^2)$. Then three cases are possible:

1) $X_2(T_*^2,u_*^2) \subset X_1(T_*^1,u_*^1) \subset X_K$,
2) $X_2(T_*^2,u_*^2) \subset X_K \subset X_1(T_*^1,u_*^1)$,
3) $X_K \subset X_2(T_*^2,u_*^2) \subset X_1(T_*^1,u_*^1)$.

We consider each of these cases and begin with the second case $X_2(T_*^2,u_*^2) \subset X_K \subset X_1(T_*^1,u_*^1)$. By (3), it is necessary to find a mixed solution $X_{ms}(\cdot,u_*)$ of the system (2) such that $X_{ms}(T_*,u_*) = X_K$.

We get the kind of desired mixed solution

$$X_{ms}(t,u_*) = \begin{cases} Y_2(t,u_*), & t \in [0,t'], \\ X_1(t,u_*), & t \in (t',T_*], \end{cases}$$

where $u_*(t) = \begin{cases} u_*^2(t), & t \in [0,t'] \\ u_*^1(t), & t \in (t',T_*] \end{cases}$, $Y_2(\cdot,u_*)$ is second basic solution of the equation, $t' = \ln\left(\frac{c+3}{2b}\right)$, $T_* = 2\ln\left(\frac{c+2}{2\sqrt{b}}\right)$ of the equation

$$D_{ps}Y = Y + u_*, \quad Y(0,u_*) = X_0,$$

$X_1(\cdot,u_*)$ is the first basic solution of the equation

$$D_{ps}X = X + u_*, \quad X(t',u_*) = Y_2(t',u_*).$$

That is, T_* is the optimal time, $u_*(\cdot)$ is the optimal control, $X_{ms}(\cdot,u_*)$ is the optimal solution.

Now consider the first option, i.e. $X_2(T_*,u_*) \subset X_1(T_*,u_*) \subset X_K$.

It's obvious that $diam(X_1(T_*,u_*)) < diam(X_K)$. However, according to the definition of the first basic solution, the function $diam(X_1(\cdot,u_*))$ is a monotonically increasing function. That is, there is such a moment time $T^* > T_*$, that $diam(X_1(T^*,u)) = diam(X_K)$. Consequently, it is necessary to construct such a control $u^*(\cdot)$, that $X_1(T^*,u^*)) = X_K$.

Since for any admissible control $u(\cdot)$ the first basic solution has the form $X_1(t,u) = e^t B_1(0) + e^t\int_0^t e^{-s}u(s)ds$, where the first term determines the form and volume of the solution at each moment of time, and the second term determines its location in space R^n. Consequently, $T^* = \ln(b)$.

Let $\tau = T^* - T_*$ and $u^*(t) = \begin{cases} u^*(t) \equiv 0, & t \in [0,\tau), \\ u^*(t) = u_*(t-\tau), & t \in [\tau,T^*]. \end{cases}$

Now we find the first basic solution $X_1(\cdot,u^*)$ of the system (2) on the interval $[0,T^*]$, that corresponds to the control $u^*(\cdot)$. Consequently, T^* is the optimal time, $u^*(\cdot)$ is the optimal control, $X_1(\cdot,u^*)$ is the optimal solution.

The third case is solved similarly. Only we build a second basic solution.

In conclusion, we make the following remarks:

1)if X_0 is not a ball (but X_0 and X_K are homothetic figures with a coefficient $k > 0$), then we describe balls around the sets X_0 and X_K and solve the problem as it was done for the balls;

2) if the behavior of the object is described by system $DX = aX + u$, $X(0,u) = X_0$, where $a \in R$, $(a \neq 0)$, $DX(t)$ is one of the previously considered derivatives $D_{ps}X(t)$ or $D_{bg}(t)$ of the set-valued mapping $X(t)$, then the problem is solved in a similar way, but Proposition 1 must be taken into account.

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