

A Two-Step Proximal Method for Equilibrium Problems in Hadamard spaces

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Abstract—We propose a novel two-step proximal method for solving equilibrium problems in Hadamard spaces. The equilibrium problem is very general in the sense that it includes as special cases many applied mathematical models such as: variational inequalities, optimization problems, saddle point problems, and Nash equilibrium point problems. The proposed algorithm is the analog of the two-step algorithm for solving the equilibrium problem in Hilbert spaces explored earlier. We prove the weak convergence of the sequence generated by the algorithm for pseudo-monotone bifunctions. Our results extend some known results in the literature for pseudo-monotone equilibrium problems.

Keywords—Hadamard space; equilibrium problem; convexity; pseudo-monotonicity; two-step proximal algorithm; convergence

I. INTRODUCTION

An equilibrium theory in finite-dimensional Euclidean spaces was first introduced by Ky Fan. Algorithms for approximate solving equilibrium and related problems are the subject of many papers [1-9]. Variational inequalities are a case of equilibrium problems [10]. G.M. Korpelevich proposed an extragradient method [11] for solving them. The analogs of the extragradient method for equilibrium problems and related questions are the subject of [7, 12].

In 1980 L.D. Popov [13] proposed an efficient interesting modification of the Arrow-Hurwitz method of search for saddle points of convex-concave functions in finite-dimensional Euclidean space. A two-step iterative proximal algorithm for solving equilibrium problems in Hilbert space, which is an adaptation of L.D. Popov's method for general equilibrium programming problems, was proposed in [8] (see also [9, 14]).

Interest in building the theory and algorithms for solving mathematical programming problems in metric Hadamard spaces [15-17] (also known as CAT(0) spaces) has arisen

recently due to problems in mathematical biology and machine learning. Another strong motivation for studying these problems is the ability to formulate some non-convex problems in the form of convex (more precisely, geodesically convex) problems in a space with a specially selected adequate Riemannian metric [18]. Some authors began to study equilibrium problems in Hadamard spaces [18-20]. In [18], existence theorems for problems of equilibrium on Hadamard manifolds were obtained, applications to variational inequalities were considered, and the resolvent method for approximating solutions to equilibrium problems was substantiated. In [19], for more general equilibrium problems with pseudo-monotone bifunctions in Hadamard spaces, existence theorems were obtained, and a proximal algorithm was proposed, and its convergence was proved. A more constructive approach is devoted to the work [20], the authors of which, starting from [7], proposed and justified an analog of the extragradient method (more precisely, extra-proximal method) for pseudo-monotone equilibrium problems in Hadamard spaces.

In this paper, which continues the article [9], we propose a novel iterative two-step proximal algorithm for approximate solution of equilibrium problems in Hadamard spaces. The algorithm is an analog of the two-step algorithms previously studied in [14] for variational inequalities and equilibrium problems in a Hilbert space or a finite-dimensional normed linear space with Bregman divergence. For pseudo-monotone bifunctions, a theorem on the weak convergence (Δ -convergence) generated by the sequence algorithm is proved.

II. PRELIMINARIES AND AUXILIARIES

In this section, we recall some well-known basic and useful results [15-17] that will be needed in establishing our main results.

Let (X, d) be a metric space. For $\xi, \eta \in X$, a mapping $m: [0, l] \rightarrow X$, where $l > 0$, is called a geodesic with endpoints ξ, η , if $m(0) = \xi$, $m(l) = \eta$, and $d(m(t), m(\tau)) = |t - \tau|$ for all $t, \tau \in [0, l]$. If, for all $\xi, \eta \in X$, a geodesic with endpoints ξ, η exists, then we call (X, d) a geodesic metric space. Furthermore, if there exists a unique geodesic for each $\xi, \eta \in X$, then (X, d) is said to be uniquely geodesic [15].

A subset K of a uniquely geodesic space (X, d) is said to be convex when for any two points $\xi, \eta \in K$, the geodesic joining ξ and η is contained in K . For each $\xi, \eta \in K$, the image of a geodesic m with endpoints ξ, η is called a geodesic segment joining ξ and η (denoted by $[\xi, \eta]$).

Let (X, d) be a uniquely geodesic metric space. For each points $\xi, \eta \in X$ and for each $t \in [0, 1]$, there exists a unique point $\zeta \in [\xi, \eta]$ such that $d(\zeta, \xi) = (1-t)d(\xi, \eta)$ and $d(\zeta, \eta) = td(\xi, \eta)$. We will use the notation $t\xi \oplus (1-t)\eta$ for denoting the unique point ζ satisfying the above statement.

Definition 1 ([15]). A geodesic space (X, d) is called CAT(0) space if for all $\xi, \eta, \zeta \in X$ and $t \in [0, 1]$ it holds that

$$d^2(t\xi \oplus (1-t)\eta, \zeta) \leq td^2(\xi, \zeta) + (1-t)d^2(\eta, \zeta) - t(1-t)d^2(\xi, \eta).$$

A complete CAT(0) space is called a Hadamard space.

Let (X, d) be a Hadamard space and (x_n) be a bounded sequence in X . Take $x \in X$. Let $r(x, (x_n)) = \overline{\lim}_{n \rightarrow \infty} d(x, x_n)$. The asymptotic radius of (x_n) is given by

$$r((x_n)) = \inf_{x \in X} r(x, (x_n))$$

and the asymptotic center of (x_n) is the set

$$a((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}.$$

It is known that in a Hadamard space, $a((x_n))$ consists exactly one point.

Definition 2 ([15]). A sequence (x_n) in a Hadamard space (X, d) Δ -converges (weakly converges) to $x \in X$ if $a((x_{n_k})) = \{x\}$, for each subsequence (x_{n_k}) of (x_n) .

Every bounded sequence in general Hadamard space has a (weak) Δ -convergent subsequence [15]. Also, every closed convex subset of a Hadamard space is Δ -closed in the sense that

it contains all Δ -lim points of every Δ -convergent subsequence [15, 16].

We present next well-known result related to the notion of Δ -convergence.

Lemma 1 ([15]). Suppose the sequence (ξ_n) of elements from a Hadamard space X converges weakly to $x \in X$. Then for all $y \in X \setminus \{x\}$ we have $\liminf_{n \rightarrow \infty} d(\xi_n, x) < \liminf_{n \rightarrow \infty} d(\xi_n, y)$.

Let (X, d) be a Hadamard space. A function $g: X \rightarrow R$ is said to be convex if $g(t\xi \oplus (1-t)\eta) \leq tg(\xi) + (1-t)g(\eta)$ for all $\xi, \eta \in X$, $t \in [0, 1]$.

III. EQUILIBRIUM PROBLEM IN HADAMARD SPACES

Let (X, d) be a Hadamard space. Take a closed and convex set $C \subseteq X$ and $F: C \times C \rightarrow R$.

The equilibrium problem $EP(F, C)$ consists of finding $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall y \in C. \quad (1)$$

The set of solutions of $EP(F, C)$ will be denoted as S .

We assume that the bifunction F satisfies the following conditions:

A1) $F(x, x) = 0$ for all $x \in C$;

A2) $F(x, \cdot): C \rightarrow R$ is convex and lower semicontinuous for all $x \in C$;

A3) $F(\cdot, y): C \rightarrow R$ is Δ -upper semicontinuous for all $y \in C$;

A4) $F: C \times C \rightarrow R$ is pseudo-monotone, i.e.

for any pair $x, y \in C$ $F(x, y) \geq 0$ implies $F(y, x) \leq 0$;

A5) $F: C \times C \rightarrow R$ is Lipschitz-type continuous, i.e. there exist two positive constants a and b such that

$$F(x, y) \leq F(x, z) + F(z, y) + ad^2(x, z) + bd^2(z, y) \quad \forall x, y, z \in C.$$

Further, we assume that the solution set S is nonempty.

A. Two-Step Proximal Algorithm

Propose the following Two-Step Proximal Algorithm for solving $EP(F, C)$.

Algorithm 1. For $x_1, y_1 \in C$ generate the sequences $x_n, y_n \in C$ with the iterative scheme

$$\begin{cases} x_{n+1} = \arg \min_{y \in C} (F(y_n, y) + \frac{1}{2\lambda} d^2(y, x_n)), \\ y_{n+1} = \arg \min_{y \in C} (F(y_n, y) + \frac{1}{2\lambda} d^2(y, x_{n+1})), \end{cases}$$

where $\lambda > 0$.

B. The Convergence of the Algorithm

We start the analysis of the weak convergence with the proof of an important inequality for sequences (x_n) and (y_n) , generated by the Two-Step Proximal Algorithm.

Lemma 2. For sequences (x_n) and (y_n) , generated by Algorithm 1 and an element $z \in S$ the next inequality is satisfied

$$d^2(x_{n+1}, z) \leq d^2(x_n, z) - (1 - 2\lambda b)d^2(x_{n+1}, y_n) - (1 - 4\lambda a)d^2(y_n, x_n) + 4\lambda ad^2(x_n, y_{n-1}). \quad (2)$$

Proof. Let $z \in S$. We have

$$F(y_n, x_{n+1}) + \frac{1}{2\lambda}d^2(x_{n+1}, x_n) \leq F(y_n, y) + \frac{1}{2\lambda}d^2(y, x_n) \quad \forall y \in C.$$

Now, letting $y = tx_{n+1} \oplus (1-t)z$, $t \in (0, 1)$, we have

$$\begin{aligned} F(y_n, x_{n+1}) + \frac{1}{2\lambda}d^2(x_{n+1}, x_n) &\leq \\ &\leq F(y_n, tx_{n+1} \oplus (1-t)z) + \\ &+ \frac{1}{2\lambda}d^2(tx_{n+1} \oplus (1-t)z, x_n) \leq \\ &\leq tF(y_n, x_{n+1}) + (1-t)F(y_n, z) + \\ &+ \frac{1}{2\lambda}(td^2(x_{n+1}, x_n) + (1-t)d^2(z, x_n) - t(1-t)d^2(x_{n+1}, z)). \end{aligned}$$

Pseudo-monotonicity of F implies that $F(y_n, z) \leq 0$. Hence,

$$\begin{aligned} (1-t)F(y_n, x_{n+1}) &\leq \\ &\leq \frac{1}{2\lambda}(- (1-t)d^2(x_{n+1}, x_n) + (1-t)d^2(z, x_n) - \\ &- t(1-t)d^2(x_{n+1}, z)). \end{aligned}$$

By letting $t \rightarrow 1$ we get

$$F(y_n, x_{n+1}) \leq \frac{1}{2\lambda}(d^2(z, x_n) - d^2(x_{n+1}, x_n) - d^2(x_{n+1}, z)).$$

From the definition of elements y_n it follows that

$$F(y_{n-1}, y_n) + \frac{1}{2\lambda}d^2(y_n, x_n) \leq F(y_{n-1}, y) + \frac{1}{2\lambda}d^2(y, x_n) \quad \forall y \in C.$$

Now, letting $y = tx_{n+1} \oplus (1-t)y_n$, $t \in (0, 1)$, we have

$$\begin{aligned} F(y_{n-1}, y_n) + \frac{1}{2\lambda}d^2(y_n, x_n) &\leq \\ &\leq F(y_{n-1}, tx_{n+1} \oplus (1-t)y_n) + \\ &+ \frac{1}{2\lambda}d^2(tx_{n+1} \oplus (1-t)y_n, x_n) \leq \\ &\leq tF(y_{n-1}, x_{n+1}) + (1-t)F(y_{n-1}, y_n) + \\ &+ \frac{1}{2\lambda}(td^2(x_{n+1}, x_n) + (1-t)d^2(y_n, x_n) - t(1-t)d^2(x_{n+1}, y_n)). \end{aligned}$$

Hence,

$$\begin{aligned} tF(y_{n-1}, y_n) - tF(y_{n-1}, x_{n+1}) &\leq \\ &\leq \frac{1}{2\lambda}(td^2(x_{n+1}, x_n) - td^2(y_n, x_n) - t(1-t)d^2(x_{n+1}, y_n)). \end{aligned}$$

By letting $t \rightarrow 0$ we get

$$F(y_{n-1}, y_n) - F(y_{n-1}, x_{n+1}) \leq$$

$$\leq \frac{1}{2\lambda}(d^2(x_{n+1}, x_n) - d^2(y_n, x_n) - d^2(x_{n+1}, y_n)).$$

We obtain

$$\begin{aligned} F(y_n, x_{n+1}) + F(y_{n-1}, y_n) - F(y_{n-1}, x_{n+1}) &\leq \\ &\leq \frac{1}{2\lambda}(d^2(z, x_n) - d^2(x_{n+1}, z) - d^2(y_n, x_n) - d^2(x_{n+1}, y_n)). \end{aligned} \quad (3)$$

Lipschitz-type continuity of bifunction F guaranties the satisfying of next inequality

$$\begin{aligned} F(y_n, x_{n+1}) + F(y_{n-1}, y_n) - F(y_{n-1}, x_{n+1}) &\geq \\ &\geq -bd^2(y_n, x_{n+1}) - ad^2(y_{n-1}, y_n). \end{aligned} \quad (4)$$

In the sequel by (3) and (4), we have

$$\begin{aligned} d^2(x_{n+1}, z) &\leq d^2(z, x_n) - d^2(y_n, x_n) - \\ &- d^2(x_{n+1}, y_n) + 2\lambda ad^2(y_{n-1}, y_n) + 2\lambda bd^2(y_n, x_{n+1}). \end{aligned}$$

The term $d^2(y_{n-1}, y_n)$ we estimate in the next way

$$d^2(y_{n-1}, y_n) \leq 2d^2(y_{n-1}, x_n) + 2d^2(x_n, y_n).$$

Taking this into account, we get the following inequality

$$\begin{aligned} d^2(x_{n+1}, z) &\leq d^2(z, x_n) - d^2(y_n, x_n) - d^2(x_{n+1}, y_n) + \\ &+ 4\lambda ad^2(y_{n-1}, x_n) + 4\lambda ad^2(x_n, y_n) + \\ &+ 2\lambda bd^2(y_n, x_{n+1}), \end{aligned}$$

i.e. the inequality (2).

Then, the main theorem holds.

Theorem 1. Assume that the bifunction F satisfies A1)–A5) and the set of solutions S is nonempty. Assume that

$$\lambda \in \left(0, \frac{1}{2(2a+b)}\right).$$

Then the sequence (x_n) generated by Algorithm 1 Δ -converges to a point of S .

C. Examples

We first recall the well-known formulation of saddle point problems in the framework of Hadamard manifolds. Then we derive on algorithm of proximal type to find the saddle point.

Let M_1 and M_2 be a Hadamard manifolds, and K_1 and K_2 the geodesic convex subset of M_1 and M_2 , respectively. A function $H : K_1 \times K_2 \rightarrow R$ is called a saddle function if $H(x, \cdot)$ is geodesic convex on K_2 for all $x \in K_1$ and $H(\cdot, y)$ is geodesic concave, i.e. $-H(\cdot, y)$ is geodesic convex on K_1 for all $y \in K_2$.

A point $\bar{z} = (\bar{x}, \bar{y}) \in K_1 \times K_2$ is said to be a saddle point of H if

$$H(x, \bar{y}) \leq H(\bar{x}, \bar{y}) \leq H(\bar{x}, y) \quad \forall (x, y) \in K_1 \times K_2.$$

The saddle point problem can be rewritten in the form of an equilibrium problem. In this case Algorithm 1 takes the next form.

Algorithm 2. For (x_1, y_1) , $(\bar{x}_1, \bar{y}_1) \in K_1 \times K_2$ generate the sequences of pairs (x_n, y_n) , (\bar{x}_n, \bar{y}_n) with the iterative scheme

$$\begin{cases} x_{n+1} = \arg \min_{x \in K_1} \left(-H(x, \bar{y}_n) + \frac{1}{2\lambda} d_1^2(x, x_n) \right), \\ y_{n+1} = \arg \min_{y \in K_2} \left(H(\bar{x}_n, y) + \frac{1}{2\lambda} d_2^2(y, y_n) \right), \\ \bar{x}_{n+1} = \arg \min_{x \in K_1} \left(-H(x, \bar{y}_n) + \frac{1}{2\lambda} d_1^2(x, x_{n+1}) \right), \\ \bar{y}_{n+1} = \arg \min_{y \in K_2} \left(H(\bar{x}_n, y) + \frac{1}{2\lambda} d_2^2(y, y_{n+1}) \right), \end{cases}$$

where $\lambda > 0$.

Let I be finite set of indices. For each $i \in I$ we are given a geodesic convex set C_i and function $H_i : C \rightarrow R$, where $C = \prod_{i \in I} C_i$. A point $\bar{x} = (\bar{x}_i)_{i \in I} \in C$ called Nash equilibrium, if for all $i \in I$ next inequalities hold

$$H_i(\bar{x}) \leq H_i(\bar{x}_i, y_i) \quad \forall y_i \in C_i.$$

Define the function $\Phi : C \times C \rightarrow R$ in such way:

$$\Phi(x, y) = \sum_{i \in I} \left(H_i(x^i, y_i) - H_i(x) \right).$$

Then a point $\bar{x} \in C$ is Nash equilibrium iff, when \bar{x} is the solution of the equilibrium problem associated to the bifunction Φ and the set C .

In this case Algorithm 1 takes the next form.

Algorithm 3. For $(x_i^1)_{i \in I}$, $(y_i^1)_{i \in I} \in C$ we generate sequences for them

$$\begin{cases} (x_i^{n+1})_{i \in I} = \arg \min_{(y_i)_{i \in I} \in C} \left(\sum_{i \in I} H_i(y_i^{i,n}, y_i) + \frac{1}{2\lambda} \sum_{i \in I} d_i^2(y_i, x_i^n) \right), \\ (y_i^{n+1})_{i \in I} = \arg \min_{(y_i)_{i \in I} \in C} \left(\sum_{i \in I} H_i(y_i^{i,n}, y_i) + \frac{1}{2\lambda} \sum_{i \in I} d_i^2(y_i, x_i^{n+1}) \right), \end{cases}$$

where $y_i^{i,n} = (y_n^j)_{j \in I, j \neq i}$, $\lambda > 0$.

IV. CONCLUSIONS

In this article, we proposed a novel iterative two-step algorithm for solving the equilibrium problems in Hadamard spaces (1). We provide the analysis and proved the weak convergence of the algorithm.

Analysis of the weak convergence of the method carried out under the assumption of the existence of solutions and under the conditions of Lipschitz-type continuity and pseudo-monotonicity for the bifunction. Finally, we give some examples where the main results can be applied.

Our results extend some known results in the literature for monotone and pseudo-monotone equilibrium problems [8, 9]

and also the related results for variational inequalities associated with pseudo-monotone operators [14].

The interesting question is the substantiation of using the Algorithm 1 as the element of the scheme of searching solution of equilibrium problem with a priori information, described in the form of inclusion to the set of fixed points of quasi-nonexpansive operator.

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