On solvability of the matrix equation AXB = C over a principal ideal domain

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Abstract—**In this paper we present conditions of solvability of the matrix equation AXB = B over a principal ideal domain. The necessary and sufficient conditions of solvability of equation AXB = B in term of the Smith normal forms and in term of the Hermite normal forms of matrices constructed in a certain way by using the coefficients of this equation are proposed. If a solution of this equation exists we propose the method for its construction.**

Keywords – matrix equation, solution, domain of principal ideal

I. Introduction

Let K denote an integer domain with an identity $e \neq 0$. Further, let $K_{m,n}$ be the set of $m \times n$ matrices over K. Denote by I_n the identity matrix of dimension n and by $0_{m,n}$ the zero $m \times n$ matrix. For any matrix $A \in K_{m,n}$ $rank\, A$ and A^t denote the rank and the transpose matrix of A respectively. We will denote by $GL(m,K)$ the set of invertible matrices in $K_{m,m}$.

Consider the matrix equation

$$AXB = C, \tag{1}$$

where $A \in K_{m,n}$, $B \in K_{k,l}$, $C \in K_{m,l}$ and X is unknown $n \times k$ matrix over K. This equation is one of the best known matrix equations in matrix theory and its applications. The problem of solvability of equation (1) has drawn the attention of many mathematicians. Many authors addressed the question when the equation (1) (over the set of real numbers $\mathbb{R}$, the set of complex number $\mathbb{C}$ or the set the quaternion skew field $\mathbb{H}$) has a solution belonging to a special class of matrices. They are given necessary and sufficient conditions (using generalized inverses) for the existence of the Hermitian, skew-Hermitian, reflexive, anti-reflexive, positive and real-positive solutions, and the general solutions. More details on this problem and many references to the original literature can be found in [1–6], [8–15], [18–22].

Many authors consider the classical systems of matrix equations over fields, commutative rings and a skew field. Mitra [13] proposed conditions for the existence of common solutions of the linear matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ over a field. In [15] conditions for the existence of a common solution of these equations over a principal ideal domain were given. Similar problems were investigated in [22] for equations over a regular ring with identity.

Let $K = F$ be a field and let A be a nonzero matrix over F. A generalized inverse of A denoted by A^- is a matrix which satisfies the equation $AA^-A = A$. It may be noted that the generalized inverse of a matrix over a commutative ring K with identity not always exists. The solvability criterion for equation (1) is written in the form. A necessary and sufficient condition for the solution of the equation $AXB = C$ is $AA^-CB^-B = C$ and in this case the general solution is $X = A^-CB^- + U + AA^-UB^-B$, where U is arbitrary $n \times k$ matrix over the field F (see [2], [13]).

On the other hand, it is well know (see [16]) that the equation (1) has a solution over a field F if and only if each of the equations $AY = C$ and $ZB = C$ has $ZB = C$ a solution. Consider the example. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ be matrices over the integer number ring $\mathbb{Z}$. It is easily verified that $Y_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the solution of the equation $AY = C$ and $Z_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is the solution of the equation $ZB = C$. Since $Y = XB$, we have $XB = Y_0$. It is easily verified that $XB = Y_0$ has no solution over ring $\mathbb{Z}$. Therefore, the solvability criterion of equation (1) cannot be transferred to rings. On the other hand, there is a little information on the solvability conditions of equation (1) over commutative rings in the literature (see [4], [15], [18], [19]).

The paper is organized as follows. In Section 2, we present necessary and sufficient conditions for solvability of the matrix equation $AXB = C$ over a principal ideal domain. In Section 3 we investigate a special case of the matrix equation $AXB = C$.

II. Main Results

Further $K = R$ is a principal ideal domain with an iden-tity element. Let $A \in R_{m,n}$ be a matrix of rank r over a prin-cipal ideal ring R. For A there exist matrices $U \in GL(m,R)$ and $V \in GL(n,R)$ such that

$$U_A A V_A = S_A = diag(a_1, a_2, \ldots, a_r, 0, \ldots, 0)$$

is a diagonal matrix, where $a_1, a_2, \ldots, a_r$ are all nonzero and $a_i \,|\, a_{i+1}$ (divides) for all $i = 1, 2, \ldots, r-1$. The matrix S_A is called the Smith normal form of the matrix A. The matrix S_A can be written in the form. $S_A = \begin{bmatrix} S(A) & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, where $S(A) = diag(a_1, a_2, \ldots, a_r) \in R_{r,r}$.

Theorem 1. *Let $A \in R_{m,n}$, $B \in R_{k,l}$, $C \in R_{m,l}$ and let*

$$S_A = U_A A V_A = diag(a_1, a_2, \ldots, a_p, 0, \ldots, 0),$$
$$S_B = U_B B V_B = diag(b_1, b_2, \ldots, b_q, 0, \ldots, 0)$$

be Smith normal forms of matrices A and B respectively, where $U_A \in GL(m,R)$, $V_A \in GL(n,R)$, $U_B \in GL(k,R)$ and $V_B \in GL(l,R)$. The matrix equation $AXB = C$ is solvable over R if and only if $U_A C V_B = \begin{bmatrix} D & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix}$, where $D \in R_{p,q}$, and $D = diag(a_1, a_2, \ldots, a_p) G \, diag(b_1, b_2, \ldots, b_q)$, where $G \in R_{p,q}$.

Proof. Let $X_0 \in M_{n,k}(R)$ be a solution of the equation $AXB = C$. From the equality $AX_0B = C$ we obtain

$$U_A A V_A V_A^{-1} X_0 U_B^{-1} U_B B V_B = U_A C V_B. \qquad (2)$$

Put $V_A^{-1} X_0 U_B^{-1} = G = \begin{bmatrix} G & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$, where $G \in R_{p,q}$ and $U_A C V_B = G = \begin{bmatrix} D & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$, where $D \in R_{p,q}$. We rewrite equality (2) in the form

$$\begin{bmatrix} S(A) & 0_{p,n-p} \\ 0_{m-p,p} & 0_{m-p,n-p} \end{bmatrix} \begin{bmatrix} G & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} S(B) & 0_{q,l-q} \\ 0_{k-q,q} & 0_{k-q,l-p} \end{bmatrix} = \begin{bmatrix} D & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$

From this we have

$D_{12} = 0_{p,l-q}$, $D_{21} = 0_{m-p,q}$, $D_{22} = 0_{m-p,l-q}$, and $diag(a_1, a_2, \ldots, a_p) G \, diag(b_1, b_2, \ldots, b_q) = D$.

Conversely, let matrices $U_A \in GL(m,R)$, $V_A \in GL(n,R)$, $U_B \in GL(k,R)$ and $V_B \in GL(l,R)$ such that

$$U_A A V_A = S_A = diag(a_1, a_2, \ldots, a_p, 0, \ldots, 0)$$

and

$$U_B B V_B = S_B = diag(b_1, b_2, \ldots, b_q, 0, \ldots, 0).$$

Further, let $U_A C V_B = \begin{bmatrix} D & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix}$, where $D = diag(a_1, a_2, \ldots, a_p) G \, diag(b_1, b_2, \ldots, b_q)$ and $G \in R_{p,q}$. From the last equality we have

$$C = U_A^{-1} \begin{bmatrix} S(A) G S(B) & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix} V_B^{-1} =$$

$$U_A^{-1} S_A V_A^{-1} V_A \begin{bmatrix} G & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix} U_A U_A^{-1} S_B V_B^{-1} =$$

$$A V_A \begin{bmatrix} G & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix} U_A B = A X_0 B,$$

where $S(A) = diag(a_1, a_2, \ldots, a_p)$, $S(B) = diag(b_1, b_2, \ldots, b_q)$ and

$$V_A \begin{bmatrix} G & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix} U_A = X_0.$$

Thus, the matrix X_0 is a solution of the matrix equation $AXB = C$ and the proof of Theorem 1 is complete.

Corollary 1. *Let the matrix equation $AXB = C$ be solvable over R. Then $S_C = S_A \Phi$ and $S_C = \Psi S_B$.*

Theorem 2. *Let $A \in R_{m,n}$, $B \in R_{k,l}$ and $C \in R_{m,l}$. Further, let $U_A \in GL(m,R)$, $V_A \in GL(n,R)$, $U_B \in GL(k,R)$ and $V_B \in GL(l,R)$ such that*

$$U_A A V_A = S_A = diag(a_1, \ldots, a_p, 0, \ldots, 0),$$
$$U_B B V_B = S_B = diag(b_1, \ldots, b_q, 0, \ldots, 0)$$

be Smith normal forms of matrices A and B respectively. If

$$U_A C V_B = \begin{bmatrix} diag(a_1, \ldots, a_p) G \, diag(b_1, \ldots, b_q) & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix},$$

where $G \in R_{p,q}$, then for arbitrary matrices $T_{12} \in R_{p,k-q}$, $T_{21} \in R_{n-p,q}$ and $T_{22} \in R_{n-p,k-q}$ the matrix

$$X_T = V_A \begin{bmatrix} G & T_{12} \\ T_{21} & T_{22} \end{bmatrix} U_B \in R_{n,k}$$

is a general solution of the equation $AXB = C$.

Proof. By Theorem 1 the matrix

$$X_0 = V_A \begin{bmatrix} G & 0_{p,k-q} \\ 0_{n-p,q} & 0_{n-p,k-q} \end{bmatrix} U_A$$

is a solution of the equation $AXB = C$. Let $T_{12} \in R_{p,k-q}$, $T_{21} \in R_{n-p,q}$ and $T_{22} \in R_{n-p,k-q}$ be arbitrary matrices over a principal ideal domain R. Consider the matrix

$$X_T = \begin{bmatrix} 0_{p,q} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

It is clear that $S_A X_T S_B = 0_{m,l}$. From this equality it follows

$$U_A^{-1} S_A X_T S_B V_B^{-1} = U_A^{-1} S_A V_A^{-1} V_A X_T U_B U_B^{-1} S_B V_B^{-1} =$$

$$A V_A X_T U_B B = A X_H B = 0_{m,l}.$$

Thus, the matrix $X_H = V_A X_T U_B = V_A \begin{bmatrix} 0_{p,q} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} U_B$ is a solution of the homogenous matrix equation $AXB = 0_{m,l}$. In this connection it should be pointed out that

$$AX_0 B + AX_T B = A(X_0 + X_T) = C.$$

Hence, the matrix $X_g = X_0 + X_T$ is a general solution of the matrix equation $AXB = C$. The proof is completed.

Let $A \in R_{m,n}$ be a non-zero matrix with $rank\ A = r$ in which the first k rows are zero, i.e., $A = \begin{bmatrix} 0_{k,n} \\ A_1 \end{bmatrix}$ and the first row of the matrix A_1 is non-zero, then, for A, there exists a matrix $W \in GL(n, R)$ such that

$$AW = H_A = \begin{bmatrix} \multicolumn{2}{c}{0_{k,n}} \\ H_1 & 0_{m_1,n-1} \\ H_2 & 0_{m_2,n-1} \\ \vdots & \vdots \\ H_r & 0_{m_r,n-r} \end{bmatrix},$$

where $H_1 = \begin{bmatrix} a_1 \\ * \end{bmatrix} \in R_{m_1,1}$, $H_2 = \begin{bmatrix} h_{21} & a_1 \\ \multicolumn{2}{c}{*} \end{bmatrix} \in D_{m_2,2}$, $\cdots$, $H_r = \begin{bmatrix} h_{r1} \ldots h_{r,r-1} & a_r \\ \multicolumn{2}{c}{*} \end{bmatrix} \in D_{m_r,r}$, and $k + m_1 + \ldots + m_r = m$.

The lower block-triangular matrix H_A is called the (right) Hermite normal form of the matrix A and it is uniquely defined for A (see [7]).

The Kronecker product of the matrices $A = \lfloor a_{ij} \rfloor \in R_{m,n}$ and $B \in R_{k,l}$ is the $mk \times nl$ matrix expressible in partitioned form as

$$A \otimes B = \begin{bmatrix} a_{11} B & \ldots & a_{1n} B \\ \vdots & \ddots & \vdots \\ a_{m1} B & \ldots & a_{m,n} B \end{bmatrix} \in R_{mk,nl}.$$

The operator vector for any matrix $C = \lfloor c_{ij} \rfloor \in R_{m,l}$ is defined in the following way (see [12, Chapter 12])

$$\mathrm{vec}(C) = [c_{11} \ldots c_{1l} \quad c_{21} \ldots c_{2l} \quad \ldots \quad c_{m1} \ldots c_{ml}]^t,$$

i.e. the entries of C are stacked columnwise forming a vector of length ml.

Let $A \in R_{m,n}$, $B \in R_{k,l}$ and $C \in R_{m,l}$. Use the Kronecker product we will write the equation $AXB = C$ as the vector equation $A \otimes B^T \mathrm{vec}(X) = \mathrm{vec}(C)$, where B^t is the transpose matrix of B (see [12, Chapter 12, Theorem 12.3.1]). Thus, applying Theorem 1 in [17] to this system of linear equations, we have the following result.

Theorem 3. *Let $A \in R_{m,n}$, $B \in R_{k,l}$ and $C \in R_{m,l}$. The matrix equation $AXB = C$ is consistent if and only if the Hermite normal forms of the matrices*

$\lfloor A \otimes B^t \quad 0_{ml,1} \rfloor$ and $\lfloor A \otimes B^t \quad \mathrm{vec}(C) \rfloor$ coincide.

Corollary 2. *Let $A_i \in R_{m,n}$, $B_i \in R_{k,l}$ and $C_i \in R_{m,l}$, $i = 1, 2$. The matrix equations $A_1 X B_1 = C_1$ and $A_2 X B_2 = C_2$ have a common solution over R if and only if the Hermite normal forms of the matrices*

$$\begin{bmatrix} A_1 \otimes B_1^t & 0_{m,1} \\ A_2 \otimes B_2^t & 0_{m,1} \end{bmatrix} \text{ and } \begin{bmatrix} A_1 \otimes B_1^t & \mathrm{vec}(C_1) \\ A_2 \otimes B_2^t & \mathrm{vec}(C_2) \end{bmatrix} \text{ coincide.}$$

III. Applications

Let $A \in R_{m,n}$ be a nonzero matrix. Special case of matrix equation (1) is the following matrix equation $AXA = A$, where X is unknown $n \times m$ matrix over R. Any solution of this equation is called generalized inverse and is denoted by A^- We note that there exist matrices over R which do not have generalized inverses. The problem when generalized inverse exists for every matrix over a commutative ring K with an identity element was study by many authors (see [4], [19] and references therein). One of the applications of theorems 1 and 2 is the following proposition.

Theorem 4. *Let $A \in R_{m,n}$ be a matrix of $rank\ A = r$. The equation $AXA = A$ has a solution over R if and only if $S_A = UAV = \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix}$, where $U \in GL(m, R)$ and $V \in GL(n, R)$.*

If equation $AXA = A$ is consistent then for arbitrary matrices $P_{12} \in R_{r,m-r}$, $P_{21} \in R_{n-r,r}$ and $P_{22} \in R_{n-r,m-r}$ the matrix $X_P = V_A \begin{bmatrix} I_r & P_{12} \\ P_{21} & P_{22} \end{bmatrix} U_B$ is its general solution.

Proof. Let $X_0 \in R_{n,m}$ be a solution of the matrix equation $AXA = A$. Further, let $U \in GL(m, R)$ and $V \in GL(n, R)$ such that $U_A A V_A = S_A = diag(a_1, \ldots, a_r, 0, \ldots, 0)$. By Theorem 1 we have $S_A V^{-1} X_0 U^{-1} S_A = S_A$.

Put $V^{-1} X_0 U^{-1} = \begin{bmatrix} D & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$, where $D \in R_{r,r}$. From the equality $S_A V^{-1} X_0 U^{-1} S_A = S_A$ we find that

$$S(A)DS(A) = S(A),$$

where $S(A) = diag(a_1, \ldots, a_r) \in R_{r,r}$ is a nonsingular matrix We can now easily show that $S(A) \in GL(r, R)$. So, we can assume that $S(A) = I_r$. Thus, $S_A = \begin{bmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$.

Conversely, if $S_A = UAV = \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix}$, where $U \in GL(n, R)$ and $V \in GL(m, R)$, then by Theorem 1 we have that the matrix $X_0 = V\begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}U$ is a solution of the equation $AXA = A$. By Theorem 2 for arbitrary matrices $P_{12} \in R_{r,m-r}$, $P_{21} \in R_{n-r,r}$ and $P_{22} \in R_{n-r,m-r}$ the matrix $X_P = V_A\begin{bmatrix} G & P_{12} \\ P_{21} & P_{22} \end{bmatrix}U_B$ is a general solution of the equation $AXA = A$. This completes the proof of Theorem 3.

Corollary 3. *Let $A \in R_{m,n}$ be a matrix with the Smith normal form $S_A = \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix}$. Then for every solution X_0 of the equation $AXA = A$ both matrices X_0A and AX_0 are idempotent matrices of rank r.*

Proof. Let a matrix X_0 be a solution of the equation $AXA = A$. From equality $AX_0A = A$ it follows that AX_0 and X_0A are nonzero matrices. Thus,

$$AX_0AX_0 = (AX_0)^2 = AX_0.$$

Similarly, $X_0AX_0A = (X_0A)^2 = X_0A$ and the proof of the Corollary is complete.

IV. Conclusions

Necessary and sufficient conditions for existence and expression of a solution of the matrix equation $AXB = C$ over a principal ideal domain are derived. Some results are true for this matrix equation over domains of elementary divisors and Bezout domains.

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