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On the semi-scalar equivalence of polynomial matrices

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***Abstract* — Polynomial matrices $A(\lambda)$ and $B(\lambda)$ of size $n \times n$ over a field F are semi-scalar equivalent if there exist a nonsingular $n \times n$ matrix P over F and an invertible $n \times n$ matrix $Q(\lambda)$ over $F[\lambda]$ such that $A(\lambda) = PB(\lambda)Q(\lambda)$. The aim of the present report is to present a triangular form of some nonsingular polynomial matrices with respect to semi-scalar equivalence.**

***Keywords* — Polynomial matrix; Equivalence of matrices; Smith normal form.**

I. Introduction

Let F be a field. Denote by $M_{n,n}(F)$ the set of $n \times n$ matrices over F and by $M_{n,n}(F[\lambda])$ the set of $n \times n$ matrices over the polynomial ring $F[\lambda]$. In what follows, I_n is the identity $n \times n$ matrix and O_n is the zero $n \times n$ matrix. A polynomial $a(\lambda) = a_0\lambda^k + a_1\lambda^{k-1} + \ldots a_k \in F[\lambda]$ is said to be monic if the first non-zero term $a_0 = 1$.

Let $A(\lambda) \in M_{n,n}(F[\lambda])$ be a nonzero matrix and $rank\, A(\lambda) = r$. For the matrix $A(\lambda)$ there exist matrices $U(\lambda), V(\lambda) \in GL(n, F[\lambda])$ such that

$U(\lambda)A(\lambda)V(\lambda) = S_A(\lambda) = diag(s_1(\lambda), s_2(\lambda), \ldots, s_r(\lambda), 0, \ldots, 0)$,

where $s_i(\lambda)$ are monic polynomials for all $i = 1, 2, \ldots, r$ and $s_1(\lambda) | s_2(\lambda) | \ldots | s_r(\lambda)$ (divides) are the invariant factors of $A(\lambda)$. The diagonal matrix $S_A(\lambda)$ is called the Smith normal form of $A(\lambda)$.

Matrices $A(\lambda), B(\lambda) \in M_{n,n}(F[\lambda])$ are said to be semi-scalar equivalent if there exist matrices $P \in GL(n, F)$ and $Q(\lambda) \in GL(n, F[\lambda])$ such that $A(\lambda) = PB(\lambda)Q(\lambda)$ (see [1], Chapter 4).

Let $A(\lambda) \in M_{n,n}(F[\lambda])$ be nonsingular matrix over an infinite field F. Then $A(\lambda)$ is semi-scalar equivalent to the lower triangular matrix [1]

$$S_l(\lambda) = \begin{bmatrix} s_1(\lambda) & 0 & \cdots & \cdots & 0 \\ s_{21}(\lambda) & s_2(\lambda) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n1}(\lambda) & s_{n2}(\lambda) & & s_{n,n-1}(\lambda) & s_n(\lambda) \end{bmatrix}$$

With the following properties:

1. $s_i(\lambda),\ i = 1, 2, \ldots, n$; are the invariant factors of $A(\lambda)$;
2. $s_i(\lambda)$ divides $s_{ji}(\lambda)$ for all $1 \le i < j \le n$.

Let $F = \{0,\ 1\}$ be a field of two elements. It is easily verified that the polynomial matrix $A(\lambda) = \begin{bmatrix} \lambda & 0 \\ \lambda^2 + 1 & (\lambda^2+1)(\lambda^2+\lambda+1) \end{bmatrix}$ over the field F is not semi-scalar equivalent to the lower triangular matrix $S_l(\lambda) = \begin{bmatrix} 1 & 0 \\ * & \lambda(\lambda^2+1)(\lambda^2+\lambda+1) \end{bmatrix}$. Thus, the triangular form $S_l(\lambda)$ for nonsingular matrices over a finite field not always exists.

It may be noted that for a singular matrix $A(\lambda)$ the matrix $S_l(\lambda)$ does not always exist.

Example. Let $F = \mathrm{R}$ be the field of real numbers. For 2×2 matrices

$$A(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^3 - 3\lambda^2 - \lambda & (\lambda^2-1)(\lambda^2-2\lambda) \end{bmatrix}$$ and

$$B(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^3 - \lambda^2 - \lambda & (\lambda^2-1)(\lambda^2-2\lambda) \end{bmatrix}$$

with entries from $\mathrm{R}[\lambda]$ there exist matrices

$$Q(\lambda) = \begin{bmatrix} 2\lambda^3 - 6\lambda^2 - 2\lambda + 9 & 2\lambda^4 - 4\lambda^3 - 2\lambda^2 + 4\lambda \\ -2\lambda^2 + 4\lambda + 4 & -2\lambda^3 + 2\lambda^2 + 2\lambda + 1 \end{bmatrix}$$

$\in GL(2, \mathrm{R}[\lambda])$ and $P = \begin{bmatrix} 1/9 & -2/9 \\ 0 & 1 \end{bmatrix} \in GL(2, \mathrm{R})$ such that $A(\lambda) = PB(\lambda)Q(\lambda)$

From this example it follows, that the triangular form $S_l(\lambda)$ is not uniquely determined for a nonsingular polynomial matrix $A(\lambda)$ with respect to semi-scalar equivalence.

Dias da Silva J.A and Laffey T.J. studied polynomial matrices up to PS-equivalence [2]. Matrices $A(\lambda), B(\lambda) \in M_{n,n}(F[\lambda])$ are PS-equivalent if $A(\lambda) = P(\lambda)B(\lambda)Q$ for some $P(\lambda) \in GL(n, F[\lambda])$ and $Q \in GL(n, F)$.

Let F be an infinite field. A nonsingular matrix $A(\lambda) \in M_{n,n}(F[\lambda])$ is PS-equivalent to the upper triangular matrix (see [2], Proposition 2)

$$S_u(\lambda) = \begin{bmatrix} s_1(\lambda) & s_{12}(\lambda) & s_{13}(\lambda) & \cdots & s_{1n}(\lambda) \\ 0 & s_2(\lambda) & s_{23}(\lambda) & \cdots & s_{2n}(\lambda) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & s_n(\lambda) \end{bmatrix}$$

with the following properties:

1. $s_i(\lambda),\ i = 1, 2, \dots, n$; are the invariant factors of $A(\lambda)$;
2. $s_i(\lambda)$ divides $s_{ij}(\lambda)$ for all $1 \le i < j \le n$;
3. if $i \ne j$ and $s_{ij}(\lambda) \ne 0$ then $s_{ij}(\lambda)$ is a monic polynomial and $\deg s_{ii}(\lambda) < \deg s_{ij}(\lambda) < \deg s_{jj}(\lambda)$.

The matrix $S_u(\lambda)$ is called a near canonical form of the matrix $A(\lambda)$ with respect to PS-equiva-lence. We note that conditions (1) and (2) for semi-scalar equivalence were proved in [1].

It is evident that matrices $A(\lambda), B(\lambda) \in M_{n,n}(F[\lambda])$ are PS-equivalent if and only if the transpose matrices $A^T(\lambda)$ and $B^T(\lambda)$ are semi-scalar equivalent. It is clear that semi-scalar equivalence and PS-equivalence represent an equivalence relation on $M_{n,n}(F[\lambda])$. On the basis of the semi-scalar equivalence of polynomial matrices in [1] algebraic methods for factorization of matrix polynomials were developed. We note that these equivalences were used in the study of the controllability of linear systems (see [3], [4]).

The semi-scalar equivalence and PS-equivalence of matrices over a field F contain the problem of similarity between two families of matrices ([1], [2], [5–7]). In most cases, these problems are involved with the classic unsolvable problem of a canonical form of a pair of matrices over a field with respect to simultaneous similarity. At present, such problems are called wild [5].

The semi-scalar equivalence of matrices includes the following two tasks: (1) the determination of a complete system of invariants and (2) the construction of a canonical form for a matrix with respect to semi-scalar equivalence. But these tasks have satisfactory solutions only in isolated cases. The ca-nonical and normal forms with respect to semi-scalar equivalence for a matrix pencil $A(\lambda) = A_0\lambda + A_1 \in M_{n,n}(F[\lambda])$ over arbitrary field F, where A_0 is nonsingular, were investigated in [8] and [9]. A canonical form with respect to semi-scalar equivalence for a polynomial matrix over a field is unknown in general case.

II. Main results

In this part we present main results of this report.

Theorem. *Let $A(\lambda) \in M_{n,n}(F[\lambda])$ be nonsingular matrix with the Smith normal form $U(\lambda)A(\lambda)V(\lambda) = S_A(\lambda) = diag\,(1, s(\lambda), \dots, s(\lambda))$, where $s(\lambda)$ is a monic polynomial and $\deg s(\lambda) = n$. The matrix $A(\lambda)$ is semi-scalar to the matrix*

$$S_l(\lambda) = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \lambda & s(\lambda) & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ \lambda^{n-2} & 0 & \cdots & 0 & s(\lambda) & 0 \\ \lambda^{n-1} & 0 & \cdots & \cdots & 0 & s(\lambda) \end{bmatrix}$$

if and only if the matrix $A(\lambda)$ admits the representation $A(\lambda) = B(\lambda)W(\lambda)$, where $W(\lambda) \in GL(n, F[\lambda])$ and $B(\lambda) = I_n\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1} \in M_{n,n}(F[\lambda])$ is a monic polynomial matrix of degree $n-1$. The matrix $S_l(\lambda)$ is uniquely defined for the matrix $A(\lambda)$.

Let $B(\lambda) \in M_{n,n}(F[\lambda])$. The matrix $B(\lambda)$ we write in the form $B(\lambda) = B_0\lambda^r + B_1\lambda^{r-1} + \ldots + B_r$, where $B_i \in M_{n,n}(F),\ i = 1, 2, \dots, n$. It is well known that a matrix polynomial equation

$$X^r B_0 + X^{r-1}B_1 + \ldots + XB_{r-1} + B_r = O_n$$

is solvable if and only if the matrix $B(\lambda)$ admits the representation $B(\lambda) = (I_n\lambda - D)C(\lambda)$, where $D \in M_{n,n}(F)$ [10]. The problem of solvability of matrix polynomial equations was investigated by many authors (see [1], [11–14] and references therein).

Following propositions gives a complete answer to the question of solvability of a matrix polynomial equation of second order over an infinite field (see also [14]).

Let $A(\lambda) = \sum_{i=0}^{r} A_i\lambda^{r-i} \in M_{2,2}(F[\lambda])$ be a nonsingular matrix. Further, let $S_l(\lambda) = \begin{bmatrix} s_1(\lambda) & 0 \\ s_{21}(\lambda) & s_2(\lambda) \end{bmatrix}$ be a near canonical form of the matrix $A(\lambda)$ with respect to semi-scalar equivalence. By [9] and based on the above, we get the following statements.

Proposition 1. *Let $s_1(\lambda) = (\lambda - \alpha_1)c_1(\lambda)$ and $s_2(\lambda) = (\lambda - \alpha_2)c_2(\lambda)$, where $\alpha_i \in F$. A matrix* *polynomial equation* $X^r A_0 + X^{r-1}A_1 + \ldots + XA_{r-1} + A_r = O_2$ *is solvable over a field* F *if and only if there exists* $\beta \in F$ *such that the matrix* $D_\beta(\lambda) = \begin{bmatrix} \lambda - \alpha_1 & 0 \\ \beta & \lambda - \alpha_2 \end{bmatrix}$ *is a left divisor of* $S_l(\lambda)$*, i. e.* $S_l(\lambda) = \mathrm{D}_\beta(\lambda)C(\lambda)$.

Proposition 2. *Let* $s_2(\lambda) = (\lambda^2 + \lambda\alpha_1 + \alpha_2)c_2(\lambda)$ *, where* $\lambda^2 + \lambda\alpha_1 + \alpha_2 \in F[\lambda]$*. A matrix polynomial equation* $X^r A_0 + X^{r-1}A_1 + \ldots + XA_{r-1} + A_r = O_2$ *is solvable over a field* F *if and only if there exists* $\delta_0, \delta_1 \in F$ *and* $\delta_0 \neq 0$ *such that the matrix* $D_\delta(\lambda) = \begin{bmatrix} 1 & 0 \\ \delta_0\lambda + \delta_1 & \lambda^2 + \lambda\alpha_1 + \alpha_2 \end{bmatrix}$ *is a left divisor of* $S_l(\lambda)$*, i. e.* $S_l(\lambda) = \mathrm{D}_\delta(\lambda)C(\lambda)$.

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REFERENCES

[1] P. S. Kazimirs'kyi. Decomposition of Matrix Polynomials into factors. Naukova Dumka, Kyiv; 1981 (in Ukrainian).

[2] J.A. Dias da Silva and T.J. Laffey. On simultaneous similarity of matrices and related questions. Linear Algebra and its applications, 291 (1999) 167–184. doi.org/10.1016/S0024-3795(98)10247-1.

[3] M. Dodig. Controllability of series connections. Electron. J. Linear Algebra, 16 (2007) 135–156. doi.org/10.13001/1081-3810.1189.

[4] M. Dodig. Eigenvalues of partially prescribed matrices. Electron. J. Linear Algebra, 17 (2008) 316–332. doi.org/10.13001/1081-3810.1266.

[5] Yu.A. Drozd. Tame and wild matrix problems. Lecture Notes in Math. 832 (1980) 242–258. doi.org/10.1007/BFb0088467.

[6] S. Friedland. Matrices: Algebra, Analysis and Applications. World Scientific; 2015.

[7] V.V. Sergeichuk. Canonical matrices for linear matrix problems. Linear algebra and its applications, 317 (2000) 53–102. doi.org/10.1016/S0024-3795(00)00150-6.

[8] V.M. Prokip. Canonical form with respect to semi-scalar equivalence for a matrix pencil with nonsingular first matrix. Ukrainian Mathematical Journal, 63(2012) 1314–1320. doi.org/10.1007/s11253-012-0580-x.

[9] V.M. Prokip. On the normal form with respect to the semi-scalar equivalence of polynomial mat-rices over the field. J. Math. Sciences, 194 (2013) 149–155. DOI:10.1007/S10958-013-1515-2.

[10] P. Lancaster and M. Tismenetsky. The theory of matrices. Second edition with applications. Academic Press, New York; 1985.

[11] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials. Academic Press, New York, 1982.

[12] V.M. Petrichkovich and V.M. Prokip. Factorization of polynomial matrices over arbitrary fields.Ukr. Math. J. 38 (1986), 409–412. doi.org/10.1007/BF01057299.

[13] E. Pereira. On solvents of matrix polynomials. Applied numerical mathematics. 47:2 (2003): 197–208. doi.org/10.1016/S0168-9274(03)00058-8.

[16] M. Slusky. Zeros of 2×2 Matrix Polynomials. Communications in Algebra, 38:11 (2010) 4212–4223. doi.10.1080/00927870903366843..