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Modeling of filtration processes in periodic porous media

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***Abstract* — Modeling of dynamic processes of diffusion and filtration of liquids in porous media are discussed. The media are formed by a large number of blocks with low permeability, and separated by a connected system of faults with high permeability. The modeling is based on solving initial boundary value problems for parabolic equations of diffusion and filtration in porous media. The structure of the media leads to the dependence of the equations on a small parameter. Assertions on the solvability and regularity of such problems and the corresponding homogenized convolution problems are considered. The statements are actual for the numerical solution of this problem with guaranteed accuracy that is necessary to model the considered processes.**

***Keywords* — homogenized problems; convolutions equations; parabolic problems; Laplace transforms.**

I. Modeling of processes in porous media

Modeling of dynamic processes of diffusion and filtration of liquids in porous media is actual when planning the use of underground resources, development of methods for preventing technogenic contamination of groundwater and the search for ways to purify such waters from contamination. Research of such processes engineering methods of observation are expensive and practically impossible, due to the need to install a large number of sensors on large territories and different depths to study the dynamics of fluid movement in a real porous environment. So the simulation is the only one the possibility of forecasting and optimization of methods for rational water extraction, purification and prevention of groundwater contamination.

In order to simulate diffusion and filtration processes in porous media, it is natural to first choose some model of such a medium. Porous media with a periodic structure are simulated most simply, since to describe such media it is sufficient to determine only the structure of the periodicity cell and then continue such a cell in a periodic manner with suitable periods. Porous periodic media formed by a large number of blocks with low permeability, and separated by a connected system of faults with high permeability will consider. It is natural to call such porous media *weakly porous.*

The homogenized equations for such weakly porous media turned out to be convolution equations, which are usually called dynamic problems *with memory effects* according to [1]. Such homogenized equations for equations depending on one or more small parameters and periodic coefficients were investigated in [2, 3, 4]. Moreover, in these papers, homogenized initial boundary value problems in convolutions were obtained, the solutions of which approximate the solutions of the original initial boundary value problem for weakly porous media, and accuracy estimates of the approximations and statements on the solvability of the homogenized problems were proved. The results were obtained under the assumption that the initial data are regular enough and the initial conditions are homogeneous. Without proving the accuracy estimates, such problems in convolutions were first established in [5] for hydrodynamic problems in porous media. Further details on hydrodynamic problems and a suitable bibliography can be found in [6].

Another approach to simulate diffusion and filtration processes in porous media is presented, for example, in [7], where statements on the two-scale convergence of solutions to solutions of two-scale homogenized problems are proved. Such two-scale problems depend on two fast and slow variables and the type of such equations is not clear. Also, the accuracy of the approximations is not clear in this case. Much more general homogenized problems were obtained in [8, 9]. However, such homogenized problems are also two-scale and contain both fast and slow variables. Further details on this approach and bibliography can also be found in [7, 10]. In the problems considered here, such variables are separated and the homogenized equations in convolutions depend only on slow variables. The same approach is also used, for example, in contemporary articles on related topics [11, 12].

The results of [2, 3, 4] were obtained under the assumption that the initial data are regular enough and the initial conditions are homogeneous. The solvability of such homogenized problems in convolutions with irregular initial data will be discussed here without homogeneity conditions.

To investigate the solvability of the homogenized problems with memory, we will use the Laplace transform method developed in [13] to study parabolic problems of general type. This method is briefly described in the third section. The formulation of diffusion and filtration problems for weakly porous media will be presented in the next section. The results presented here are partially announced in [14].

II. Periodic porous media

In order to determine the periodic porous media, we will consider a partition of the entire space $\mathbb{R}^3$ into two open sets E_1^ε and E_0^ε separated by boundary ∂E_1^ε. Thus, $\mathbb{R}^3 = E_1^\varepsilon \cup E_0^\varepsilon \cup \partial E_1^\varepsilon$, where ε denotes some parameter. It is assumed that the sets are ε-periodic (with a period ε in each of the independent variables x_1, x_2, x_3) and E_1^ε is connected set with the locally Lipschitz boundary ∂E_1^ε. For $\varepsilon = 1$, the sets E_1^1 and E_0^1 are completely determined by the sets $Y_1 = E_1^1 \cap Y$ and $Y_0 = E_0^1 \cap Y$ with the boundary ∂Y_1, where $Y = (0,1)^3$ denotes a cell of periodicity. It is assumed that the sets Y_1 and Y_0 have positive Lebesgue measures in $\mathbb{R}^3$.

Let us fix an open domain $\Omega \subset \mathbb{R}^3$. Then, for some sufficiently small parameter ε, the sets E_1^ε and E_0^ε naturally define periodic porous media by the equalities

$$\Omega_0^\varepsilon = E_0^\varepsilon \cap \Omega \quad and \quad \Omega_1^\varepsilon = E_1^\varepsilon \cap \Omega.$$

We will use the permeability tensor for these media given through the following definition

$$D^\varepsilon = \varepsilon^2 D_0 \quad in \quad \Omega_0^\varepsilon, \qquad D^\varepsilon = D_1 \quad in \quad \Omega_1^\varepsilon, \qquad (1)$$

where the constant matrices D_0 and D_1 are symmetric and elliptic. The components of these matrices characterize the permeability properties of the media under consideration. Using these definitions, we define a function $u = u(t,x)$ as a solution to the following initial boundary value problem

$$u_t' - div D^\varepsilon(\nabla u + g) = f \quad in \quad \Omega \times (0,\infty), \qquad (2)$$
$$u|_{t=0} = u_0 \ in\ \Omega, \quad u = 0\ on\ \partial\Omega \times (0,\infty),$$

which depends on the small parameter ε according to (1) and $g \in L^2(0,\infty; L^2(\Omega)^3)$, $f \in L^2(0,\infty; H^{-1}(\Omega))$, and $u_0 \in L^2(\Omega)$ are given functions that simulate external influences. Here and below, function spaces are used, the definitions of which are given, for example, in [1].

Therefore, for small ε, the equation of problem (2) degenerates on the set that simulates the blocks with very low penetration. This dependence on a small parameter leads to the homogenized problem in convolutions, the solutions of which approximate the solution of problem (2) for small ε in accordance with [2, 4]. For a precise formulation of such homogenized problems, additional definitions are needed.

Let the function $q = q(t,y)$ be a 1-periodic solution of the initial boundary value problem:

$$q_t' - div_y\big(D_0 \nabla_y q\big) = 0 \quad in \quad Y_0 \times (0,\infty), \qquad (3)$$
$$q|_{t=0} = 1 \ in\ Y_0, \quad q = 0\ on\ \partial Y_0 \times (0,\infty).$$

It is known [1] that a suitable solution to this problem exists and the following function

$$r(t) = |Y_1|^{-1} \int_{Y_0} q_t'(t,y)\,dy, \qquad (4)$$

is well defined as an element of the space $L^1(0,\infty)$ in accordance with [4]. Here $|Y_1|$ denotes the Lebesgue measure of the set Y_1. Also, following [2, 4], one can determine the constant real matrix D, which characterizes the homogenized (averaged) permeability for the considered medium Ω_1^ε.

In such definitions and (4), the homogenized (averaged) problem for (2) is convolution problem for the function $v = v(t,x)$ of the following form

$$v_t' - r*(v_t') - div D(\nabla v + g) = f - r*f \quad in\ \Omega \times (0,\infty), \ (5)$$
$$v|_{t=0} = u_0 \ in\ \Omega, \quad v = 0\ on\ \partial\Omega \times (0,\infty),$$

where * denotes the convolution operator by t, for example, we have

$$r*(v_t') = \int_0^t r(t-\tau)(v_\tau'(\tau,x))\,d\tau.$$

For fixed ε, a unique solution to problem (2) exists, for example, according to [1]. For sufficiently smooth data and $u_0 = 0$, a unique solution to problem (5) exists in accordance with [4]. Moreover, the solution of problem (5) approximates the solution of problem (2) in the appropriate sense [2, 4] for small ε. More precisely, the following statement is fulfilled for the solutions.

Theorem 1. *Assume that* $g \in C_0^\infty([0,\infty] \times \Omega)^3$, $f \in C_0^\infty([0,\infty] \times \Omega)$, *and* $u_0 = 0$. *Let* T *be fixed and* $u = u(t,x)$ *be a solution to (2) and* $v = v(t,x)$ *be a solution to (5). Then*

$$\| u - v - q_\varepsilon * f + q_\varepsilon * (v_t') \|^2_{C^0([0,T];L^2(\Omega))} \le C\varepsilon,$$
$$\| u - v \|^2_{C^0([0,T];L^2(\Omega_1^\varepsilon))} \le C\varepsilon,$$

where $q_\varepsilon = q_\varepsilon(t, x/\varepsilon)$ *and the constant* C *does not depend on the small parameter* ε.

Thus, instead of solving problem (2), it is possible to solve problem (5) with a guaranteed accuracy. Naturally, the numerical solution of problem (2) for very small ε is much more complicated than the numerical solution of problem (5), since a very fine mesh is required, taking into account the shape of small blocks and faults for the media. The estimates of Theorem 1 are also valid for irregular initial data, for example, through approximation by smooth data. But, the exact formulation is much more complicated.

In addition, according to the estimates of Theorem 1, the solution to the original problem (2) is strongly oscillating on blocks, which should also be displayed in the numerical solution of this problem. It is these oscillations that lead to the appearance of convolutions in the homogenized problem, which is also commonly called the *appearance of memory* in porous media.

Thus, the presence of weakly porous blocks in the domain Ω is modeled by the appearance of a memory in density (coefficient at the time derivative) in the homogenized medium. Here we will investigate the solvability and regularity for problem (5) with common initial data, since it is necessary for the numerical solution of this problem with guaranteed accuracy.

The main result on the solvability and regularity for the problem is the following statement.

Theorem 2. *For every* $g \in L^2(0,\infty;L^2(\Omega)^3)$, $f \in L^2(0,\infty;H^{-1}(\Omega))$, *and* $u_0 \in L^2(\Omega)$ *there exists the unique solution* $v \in L^2(0,\infty;H_0^1(\Omega))$ *to problem (5) and there is a positive constant* C, *such that*

$$\| v \|_{L^2(0,\infty;H_0^1(\Omega))} + \| v_t' \|_{L^2(0,\infty;H^{-1}(\Omega))} \le$$
$$\le C\| g \|_{L^2(0,\infty;L^2(\Omega)^3)} + C\| f \|_{L^2(0,\infty;H^{-1}(\Omega))} + C\| u_0 \|_{L^2(\Omega)}$$

and $v \in C^0([0,T];L^2(\Omega))$ *for fixed positive* T.

III. Laplace transform and a priori estimates

We define the space $L^2_\omega(0,\infty;L^2(\Omega))$ for a fixed real ω as the function set from the space $L^2_{loc}(0,\infty;L^2(\Omega))$, for which the quantity $\| u \|_{L^2_\omega(0,\infty;L^2(\Omega))} = \| e^{-\omega t} u \|_{L^2(0,\infty;L^2(\Omega))}$ is finite. The last equality defines a norm in the space $L^2_\omega(0,\infty;L^2(\Omega))$, with respect to which this space is complete in accordance with [13]. Let the space $E_\omega(L^2(\Omega))$ be the set of functions $W = W(\sigma)$ with values in $L^2(\Omega)$, continuous and holomorphic in the half-plane $\mathbb{C}_\omega = \{\sigma \in \mathbb{C} : \sigma = \sigma_1 + i\sigma_2, \sigma_1 > \omega\}$, for which the quantity

$$\| U \|^2_{E_\omega(L^2(\Omega))} = \int_{-\infty}^{\infty} \| U(\omega + i\sigma_2) \|^2_{L^2(\Omega)} d\sigma_2$$

is finite. The equality defines the norm in $E_\omega(L^2(\Omega))$.

It is known [13] that the Laplace transform

$$\tilde{w}(t) = \int_0^\infty e^{-\sigma t} w(t) dt = W(\sigma)$$

is a bijective bicontinuous map from $L^2_\omega(0,\infty;L^2(\Omega))$ into $E_\omega(L^2(\Omega))$ for a fixed real ω.

We will denote $V = \hat{v}$, $R = \hat{r}$, $Q = \hat{q}$, $G = \tilde{g}$, and $F = \tilde{f}$. Applying the Laplace transform to (5), we get

$$\sigma(1-R)V - divD(\nabla V) = \mathbb{F} \quad in \quad \Omega, \quad V|_{\partial\Omega} = 0, \quad (6)$$

where $\mathbb{F} = divDG + F(1-R) + u_0(1-R)$, since the Laplace transform maps the convolution operator into pointwise multiplication.

For fixed $\sigma \in \mathbb{C}$, problem (6) is a boundary value problem for an elliptic equation with complex coefficients in the lower order terms. It is known [13], that the problem is solvable for all $\sigma \in \mathbb{C}$ except, perhaps, a discrete set in $\mathbb{C}$. Here, in order to explain the solvability of problem (5), it will be enough to separate from this set using a priori estimates with constants independent of $\sigma \in \mathbb{C}_0$.

In order to do this, we multiply equation (6) by $\overline{V}$ and integrate over Ω. Then, we get

$$\sigma(1-R)\int_\Omega |V|^2 dx + \int_\Omega (D\nabla V, \nabla\overline{V}) dx = \int_\Omega \mathbb{F}\overline{V} dx.$$

Similarly, multiplying the equation conjugate to (6) by V and integrating over Ω, we have

$$\overline{\sigma}(1-\overline{R})\int_\Omega |V|^2 dx + \int_\Omega (D\nabla\overline{V}, \nabla V) dx = \int_\Omega \overline{\mathbb{F}} V dx.$$

Consequently, using the summation and the ellipticity of the homogenized matrix D, we conclude that

$$(\sigma_1 + Re(-\sigma R)) \| V \|^2_{L^2(\Omega)} + \alpha \| V \|^2_{H_0^1(\Omega)} \le Re \int_\Omega \mathbb{F}\overline{V} dx,$$

where α denotes the ellipticity constant of matrix D.

It is possible to check that the function $Re(-\sigma R(\sigma))$ is non-negative and the function $R(\sigma)$ is bounded for $\sigma \in \mathbb{C}_0$. Thus, it follows from the last inequality that

$$\alpha \| V \|^2_{H_0^1(\Omega)} \le Re \int_\Omega \mathbb{F}\overline{V} dx \le \| \mathbb{F} \|_{H^{-1}(\Omega)} \| V \|_{H_0^1(\Omega)} .$$

Therefore, using the definition of $\mathbb{F}$ from (6), we get for solution (6) the following a priori estimate

$$\| V \|_{H_0^1(\Omega)} \le C \| g \|_{L^2(\Omega)^3} + C \| f \|_{H^{-1}(\Omega)} + C \| u_0 \|_{H^{-1}(\Omega)} \quad (7)$$

with a constant C, which is independent of $\sigma \in \mathbb{C}_0$.

It follows from the obtained inequality that there exists the unique solution $V \in H_0^1(\Omega)$ to problem (6) for every $\sigma \in \mathbb{C}_0$. The solution has some additional properties. Namely, following [4], one can prove that the solution is continuous and holomorphic on $\mathbb{C}_0$.

Indeed, let us check, for example, the continuity of this solution. To do this, fix $\upsilon \in \mathbb{C}_0$ and assume that $\sigma \to \upsilon$. For simplicity, we introduce the notation

$$S(\sigma) = \sigma(1 - R(\sigma)).$$

Then problems (6) at points σ and υ have the form

$$-divD(\nabla V(\sigma)) + S(\sigma)V(\sigma) = \mathbb{F}(\sigma), \quad V(\sigma)|_{\partial\Omega} = 0,$$
$$-divD(\nabla V(\upsilon)) + S(\upsilon)V(\upsilon) = \mathbb{F}(\upsilon), \quad V(\upsilon)|_{\partial\Omega} = 0.$$

Using that $S(\upsilon) = S(\sigma) + (S(\upsilon) - S(\sigma))$ and subtraction, we get for $W(\sigma) = V(\sigma) - V(\upsilon)$ the problem

$$- divD(\nabla W(\sigma)) + S(\sigma)W(\sigma) = \mathbb{F}(\sigma) - \mathbb{F}(\upsilon) + \\ + (S(\upsilon) - S(\sigma))V(\upsilon), \quad W(\sigma)|_{\partial\Omega} = 0.$$

Thus, repeating the proof of (7), we conclude that

$$\| W \|_{H_0^1(\Omega)} \le C \| \mathbb{F}(\sigma) - \mathbb{F}(\upsilon) \|_{H^{-1}(\Omega)} + \\ + | S(\upsilon) - S(\sigma) | C \| V(\upsilon) \|_{H^{-1}(\Omega)} \to 0$$

as $\sigma \to \upsilon$, since $\mathbb{F}(\sigma)$ and $S(\sigma)$ are continuous, which follows from the definitions and known properties of solutions to equation (3), for example, according to [1].

Therefore, using the bijective bicontinuous map from $E_0(H_0^1(\Omega))$ into $L_0^2(0,\infty;H_0^1(\Omega))$, we can derive the estimate of Theorem 2 from equality (7). Thus, using the well-known embedding theorem given, for example, in [1], we conclude for solution to problem (5) that the inclusion $v \in C^0([0,T];L^2(\Omega))$ is valid.

IV. Conclusion

Thus, initial-boundary value problems for non-stationary equations of diffusion and filtration in weakly porous media were discussed. Assertions on the solvability and regularity of such problems and the corresponding homogenized convolution problems with memory have been submitted. These statements are presented for general initial data and inhomogeneous initial conditions and generalize classical results on the solvability of initial boundary value problems for the heat equation. The proofs use the methods of a priori estimates and the well-known Agranovich-Vishik method based on the Laplace transform and developed to study parabolic problems of general type. The statements are necessary for the numerical solution of this problem with guaranteed accuracy.

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