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Generalized solvability and optimal control for an integro-differential equation of a hyperbolic type

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***Abstract* — We consider an integro-differential operator with Volterra type integral term. We provide a priory inequalities in negative norms for certain spaces. Further, using obtained inequalities we prove well-posedness (existence and uniqueness of the (weak) generalized solution) of the corresponding boundary value problem as well as a theorem on optimal control existence**

***Keywords* — Integro-differential equation; generalized solvability; optimal control; a priory inequalities; Volterra operator.**

I. INTRODUCTION

The equations of hyperbolic type are one of the well-known and extensively studied PDEs. Mostly due to the significance of the wave equation. On the other hand, partial integro-differential equations (PIDE) could be more appropriate for simulating physical processes. For example, Volterra integro-differential equations describe various processes in materials with memory [1], [2], [3]. The latter include, for example, some polymers and concrete mixtures.

In this paper, we consider a partial integro-differential equation that generalizes the wave equation. Its right-hand side belongs to some negative space. This includes (among others) impulse, pointwise, and other actions on the system (see [4]).

Using the method of a priori inequalities in negative spaces [4], [5], [6] we show that there exists a unique weak solution of the equation and optimal control for the corresponding system.

Main notations and functional spaces

In the cylindrical domain $Q=\Omega\times(0,T)$, we consider a system described by the linear integro-differential equation

$$Lu=\frac{\partial^2 u}{\partial t^2}+Au+Bu=F,$$

where $\Omega\subset R^n$ is a bounded connected domain of the space variables with regular boundary $\partial\Omega$. Here

$$Au=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\sum_{i=1}^{n}a_i(x)\frac{\partial u}{\partial x_i}+a(x)u,$$

$$Bu=\int_0^t\sum_{i=1}^{n}K_i(t,\tau)u_{x_ix_i}(x,\tau)\,d\tau.$$

In the paper, we suppose that the kernels $K_i(t,\tau)$ are continuous in $[0,T]^2$ and have a continuous derivative with respect to τ. Furthermore, let a_{ij}, a_i, a be continuous functions in $\overline{\Omega}$, $a_{ij}=a_{ji}$ and there exists positive number α such that $\sum_{i,j=1}^{n}a_{ij}(x)\lambda_i\lambda_j\geq\alpha\sum_{i=1}^{n}\lambda_i^2$, for all $\lambda_i\in R$ and $x\in\overline{\Omega}$.

Function $u(t,x)$ satisfies the following boundary and initial conditions

$$u\,|_{t=0}=\frac{\partial u}{\partial t}\,|_{t=0}=0,\quad u\,|_{\partial\Omega}=0.$$

By L we denote the set of all functions $u\in C^\infty(\overline{\Omega})$ such that

$$u\,|_{t=0}=\frac{\partial u}{\partial t}\,|_{t=0}=\ldots=0,$$

and by L_T we denote the set of all functions $u\in C^\infty(\overline{\Omega})$ such that

$$u\,|_{t=T}=\frac{\partial u}{\partial t}\,|_{t=T}=\ldots=0.$$

By H_0^k, S_0^k, V_0^k, H_T^k, S_T^k, V_T^k we denote the completion of the sets L, L_T with respect to the norms

$$\|u\|_{H_0^k}^2=\int_Q(u^{(k)})^2+\sum_{i=1}^{n}(u_{x_i}^{(k-1)})^2\,dQ,$$

$$\|u\|_{V_0^k}^2=\|u\|_{H_0^k}^2+\sum_{i=1}^{n}\int_\Omega(u_{x_i}^{(k-1)})^2\,|_{t=T}\,d\Omega,$$

$$\|u\|_{S_0^k}^2=\|u\|_{V_0^k}^2+\int_\Omega(u^{(k)})^2\,|_{t=T}\,d\Omega,$$

$$\|v\|_{H_T^k}^2=\int_Q(v^{(k)})^2+\sum_{i=1}^{n}(v_{x_i}^{(k-1)})^2\,dQ,$$

$$\|v\|_{V_T^k}^2=\|v\|_{H_T^k}^2+\sum_{i=1}^{n}\int_\Omega(v_{x_i}^{(k-1)})^2\,|_{t=0}\,d\Omega,$$

$$\|v\|_{S_T^k}^2=\|v\|_{V_T^k}^2+\int_\Omega(v^{(k)})^2\,|_{t=0}\,d\Omega,$$

respectively. Here $u^{(k)}$ means a derivative of order k with respect to the variable t.

II. RELATED WORKS

There are a lot of papers that use the method of a priori inequalities in negative spaces for various BVP for PDE. See, for example, [7], [8], [9], [10] and the bibliography there. This approach is also appropriate for PIDE. For example, equations of parabolic type were considered in [11], a problem with a non-negative definite integral operator was considered in [12].

In the paper [13] authors consider the case of a purely differential equation ($Bu=0$) and obtain a priory inequalities for operator L and some results on weak solvability with any integer k. Further, in [14] the case of integro-deferential equation is considered. In case $k=1$ (in the triple S^0, V^1, H^1) results on weak solvability are obtained. Finally, in [15] authors consider triple S^1, V^0, H^2 (that corresponds to $k=2$) and provide theorems of generalized solvability. The main goal of the presented paper is to provide a priory inequalities and weak solvability theorems in case $k=3$, namely in the triple S^2, V^{-1}, H^3 .

III. Proposed technique

We claim that the following two estimations hold.

Lemma 1. There exists a positive number c such that the inequality

$$\| Lu \|_{(V_T^{-1})^*} \le c \| u \|_{H_0^3}$$

holds for every function $u \in L_0$.

Using the latest lemma we extend operator L onto the entire space H_0^3.

Lemma 2. There exists a positive number c such that the inequality

$$c^{-1} \| u \|_{S_0^2} \le \| Lu \|_{(V_T^{-1})^*}$$

holds for every function $u \in H_0^3$.

Now, let us consider a problem

$$Lu = F,\, F \in (V_T^{-1})^*.$$

Definition. The function $u \in H_0^3$ is said to be a generalized solution of the problem $Lu = F, F \in (V_T^{-1})^*$ if there exists a sequence of functions $u_i(x,t) \in L_0$ such that

$$\| u - u_i \|_{S_0^2} \to 0, \quad \| Lu_i - F \|_{(V_T^{-1})^*} \to 0, i \to \infty.$$

Using the approach from [4] we can prove the theorems on generalized solvability, optimal control, provide a numerical method for mentioned problem solving and prove the convergence theorem.

In particular, we consider the optimal control problem

$$Lu = f + C(h),$$
$$J(h) \to \min.$$

Here h is a control from an admissible set $U_\partial \subseteq H.$ Let the operator C has the following form

$$C(h) = \sum_{i=1}^{s} \delta(t - t_i) \otimes \phi_i(x),\, h = \{(t_i, \phi_i)\}_{i=1}^{s}.$$

In this case $H = (\square \times L_2)^s$ is the corresponding control space.

IV. Results/Discussions

Theorem 1. For every $F \in (V_T^{-1})^*$ there exists the unique generalized solution for the problem $Lu = F$.

Theorem 2. There exists positive number c such that the inequality $\| u \|_{H_0^3} \le c \| F \|_{(V_T^{-1})^*}$, holds for every $F \in (V_T^{-1})^*$. Here u is the generalized solution for the problem $Lu = F$.

Theorem 3. Assume that the set of admissible control $U_\partial \subseteq H$ is closed, bounded and convex in the space H . Moreover, let $J(u) = \Phi(u(h))$ be lower semi-continuous with respect to u . Then there exists an optimal control for the problem

$$Lu = f + C(h),\, J(h) \to \min.$$

Remark. The claim of the theorem remains true for other weakly continuous operators of control C as well.

V. Conclusion

We have proved the so-called well-posedness of the problem. Using the proved a priory estimates and utilizing approaches from [4], [6] we further considered an optimal control problem and provided the theorem of optimal control existence. Further, it is possible to construct a numerical method for evaluating the generalized solution and mentioned optimal control, etc. We would like to mention as well, that cases $k \ge 4$ are still to be considered as far as it requires non-trivial choosing of so-called "test functions" while establishing a priory inequalities.

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