Optimal control of a hyperbolic system that describes Slutsky demand

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***Abstract* — A non-linear optimal control problem for a hyperbolic system of first order equations on a line in the case of degeneracy of the initial condition line is considered. This problem describes many natural, economic and social processes, in particular, the optimality of the Slutsky demand, the theory of bio-population, etc. The research is based on the method of characteristics and the use of nonstandard variations of the increment of target functional, which leads to the construction of efficient computational algorithms.**

***Keywords* — optimal control; hyperbolic equations; Slutsky equations; curved sector.**

I. Introduction

The paper [2] describes consumer behaviour from a consumption theory which mathematical model is reduced to finding an optimal solution of the Cauchy problem for a hyperbolic system of first order equations with two independent variables. (Slutsky demand response to a change in price and demand.) An extended version of such a model with boundary conditions is considered in [4].

We propose a variant of the optimal control problem, for the case of a domain where the initial conditions line degenerates into a point. From the consumption theory it means that the degenerate point is zero consumption factor and after $t = T$ there is no increase in the benefit of consumption.

II. Statement of the problem

Hence, to find the level of Slutsky demand under increasing prices and capital in terms of the established notation of the theory of hyperbolic systems of first order equations with two independent variables we formulate the problem: in the domain

$(x,t) \in S = \{x,\ t : a(t) < x < b(t),\ 0 < t < T,\ a(0) = b(0) = 0\}$ we consider some process $y = y(x,t)$, which evolution in time and space we describe by semilinear system of first order hyperbolic equations

$$\frac{\partial y(x,t)}{\partial t} + \lambda(x,t)\frac{\partial y(x,t)}{\partial x} = f(y(x,t),x,t), \qquad (1)$$

where $y : S \to \mathbb{R}^n$ is the vector-function of solution, λ is the reflection from $\bar{S}$ on the space of $n \times n$ diagonal real-valued matrices

$$\lambda(x,t) = diag(\lambda_1(x,t), \lambda_2(x,t), \ldots, \lambda_n(x,t)),$$

$f : \mathbb{R}^n \times S \to \mathbb{R}^n$ is the given nonlinear vector-function and $a : [0,T] \to \mathbb{R}^1$, $b : [0,T] \to \mathbb{R}^1$.

Note that in the one-dimensional case an arbitrary semilinear hyperbolic system with a nondiagonal characteristic matrix can always be reduced to a semilinear hyperbolic system with a diagonal matrix [3].

Let us consider sets

$$I = \{1, 2, \ldots, n\},$$

$$I_a = \{i \in I,\ \lambda_i(x,t) > 0,\ (x,t) \in \bar{S}\},$$

$$I_b = \{i \in I,\ \lambda_i(x,t) < 0,\ (x,t) \in \bar{S}\},$$

for which $m_1 = card(I_a)$, $m_2 = card(I_a)$.

That is, we assume that the first m_1 eigenvalues of the matrix $\lambda(x,t)$ are positive and the remaining $m_2 = n - m_1$ are negative. This distribution of signs of eigenvalues implies that none of the characteristics of equation (1) that exit from the point $(0,0)$ in the direction of increasing t, fall into $\bar{S}$.

For system (1), let us set nonlinear boundary conditions

$$y_+(a(t),t) = \gamma^a(y_-(a(t),t), u^{(1)}(t), t),\ t \in [0,T], \quad (2)$$

$$y_-(b(t),t) = \gamma^b(y_+(b(t),t), u^{(2)}(t), t),\ t \in [0,T]. \quad (3)$$

Here $u^{(1)}$, $u^{(2)}$ are the controlling influences, such that for the compacts U^1, U^2, $u^{(i)} : [0,T] \to U^i$, $U^i \subset \mathbb{R}^{r_i}$ $(r_i \in \mathbb{N},\ i = 1,2)$; $y_+ : \bar{S} \to \mathbb{R}^{m_1}$,

$y_- : \overline{S} \to \mathbb{R}^{n-m_1}$ are the vector y subvectors that correspond to positive and negative eigenvalues of the characteristic matrix of the system (1) (we will use analogous notation for the other functions);

$$\gamma^a : \mathbb{R}^{n-m_1} \times U^1 \times [0,T] \to \mathbb{R}^{m_1},$$

$$\gamma^b : \mathbb{R}^{m_1} \times U^2 \times [0,T] \to \mathbb{R}^{n-m_1}.$$

The target functional has the form

$$J(u^{(1)},u^{(2)}) = \int_0^T G_0(y_-(a(t), y_+(b(t),t),t))dt +$$

$$+\iint_S G(y(x,t),x,t)dxdt, \quad (4)$$

where $G_0 : \mathbb{R}^n \times [0,T] \to [0,T]$; $G : \mathbb{R}^n \times \overline{S} \to \overline{S}$

and these functions are measurable on $[0,T]$ for arbitrary function y from the corresponding space. So, we should investigate the problem

$$\min_{u^{(1)},u^{(2)}} J(u^{(1)},u^{(2)}), \quad (5)$$

where the minimum is taken for those $u^{(1)}$, $u^{(2)}$, for which there exists a single solution of the problem (1)–(3).

In this paper we prove the correct solvability of the formulated problem under the satisfying of the proper smoothness conditions for initial data of the problem (1) – (4) and under the fulfilling of the conditions of the agreement

$$\gamma^a(y_-(0,0),u^{(1)}(0),0) = \gamma^b(y_+(0,0),u^{(2)}(0),0).$$

The variation analysis of the investigated problem is constructed by the rule

$$u^i_{\varepsilon,\delta}(t) = u^{(i)}(t+\varepsilon\delta^{(i)}(t)),\ t\in[0,T],\ i=1,2, \quad (6)$$

where $\varepsilon \in [0,1]$ is the parameter characterizing smallness of variation, $\delta^{(i)}(t)$ is continuously differentiable function, which satisfies condition

$$0 \le t+\delta(t) \le T,\ t\in[0,T],\ \delta(0)=\delta(T)=0.$$

Choosing variation by rule (6) we found necessary optimality conditions for problem (1) – (5), which are formulated as appropriate theorems.

In simplified such models, for example, for the optimal control problems of age-structured population or in the problems of renewal of gravity wave profile, effective numerical algorithms for solving corresponding optimal control problems were constructed [1].

Remark. If characteristics of the system (1), that exit from the point $(0,0)$ fall in the domain S, then the boundary conditions (2) – (3) have another number, see, for example, [5].

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