Operator for Construction of Archimedian Triangular Norms

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Abstract— **Triangular norms and associative functions are base of connectives in fuzzy logic and fuzzy systems. New connective operator that can generate different classes of fuzzy connectives is proposed. It is proved that this operator satisfies the requirements of such axioms as commutativity, associativity, monotonicity and boundary conditions. It is parameterized and therefore new triangular norms are obtained. Constructed parameterized triangular norms are of a strict and Archimedian type.**

Keywords — triangular norms, fuzzy logic, logical connectives, connective generators

I. Introduction

Developing associative functions, Menger [1] introduced the concept of a triangular norm for the generalization of the inequality of a triangle in a certain space. Schweizer and Sclar introduced the modern definition of the triangular norm in the theory of probabilistic metric spaces [2]. These norms can be used as a generalization of the connectives of Boolean logic to many-valued and fuzzy logic [3-5]. This fact has led to an increase in the interest and comprehensive development of the theory of triangular norms [6, 7]. Nowadays fuzzy sets theory and fuzzy logic are the basis for modern systems of management and decision-making in many branches of industry. Therefore, creating methods for building of new triangular norms is an actual task.

This paper is organized as follows. At the beginnings in Section II we present known operators for construction of Archimedian triangular norms. New operator will be proposed in Section III. In Section IV new obtained triangular norms will be shown.

II. Existing operators for triangular norms construction

Aczel [1] proposed conventional generator for triangular *t*-norm construction which is expressed as

$$h_1(x,y) = f^{-1}(f(x)+f(y)), \qquad (1)$$

where $x \in [0,1]$, $f(x) \in [0,\infty)$, $\lim_{x\to 0} f(x) = \infty$, $f(1)=0$, $f(\cdot)$ is monotonically decreasing function and $f^{-1}(\cdot)$ – inverse to $f(\cdot)$ function which is known as a pseudoinverse.

A new class of generators is described in [8, 9]. It is based on the use of the next operator

$$h_2(x,y) = f(f^{-1}(x)+f^{-1}(y)+f^{-1}(x)f^{-1}(y)), \qquad (2)$$

where $x \in [0,\infty)$; $f(x) \in (0,1]$; $f(0)=1$; $\lim_{x\to\infty} f(x) = 0$, $f^{-1}(x) \in (0,1]$.

However, the disadvantage of such an operator (2) is the lack of direct parameterization of the obtained by it functions which would help to control the change of their properties. To eliminate this drawback, we have chosen as a basis an operator of a different kind than (2). It is associated with the solution of the functional equations of the form [8]

$$f(x+y) = f(x)+f(y)+f(x)f(y) \qquad (3)$$

and which, according to the description (3), takes the form

$$h_3(x,y) = f^{-1}[f(x)+f(y)+f(x)f(y)], \qquad (4)$$

where $x \in [0,1)$, $\lim_{x\to 0} f(x) = \infty$, $f(1)=0$.

The another two types of operators for generating triangular norms are proposed in [10]:

$$h_4(x,y) = \varphi^{-1}\left(\frac{\varphi(x)\varphi(y)}{\varphi(x)+\varphi(y)}\right) = h_4(y,x) \qquad (5)$$

and

$$h_5(x,y) = \varphi^{-1}\left(\frac{\varphi(x)\varphi(y)}{1+\varphi(x)+\varphi(y)}\right) = h_5(y,x). \qquad (6)$$

This generators will create a *t*-norms when $\varphi(x): [0,\infty) \to (0,1]$, and $\varphi^{-1}(x): (0,1] \to [0,\infty)$.

However, operators (4)-(6) are also characterized by the same disadvantage - the lack of direct parameterization of the obtained triangular norms. Therefore, in [11], generator of another type is proposed

$$h_6(x,y) = f^{-1}[f(x)+f(y)+pf(x)f(y)], \qquad (7)$$

where $p > 0$. To expand the ability to construct new triangular norms, the new operator will be considered in the next section. This operator develops and generalizes known operators (5) and (6) via its parameterization.

III. Construction of the New Operator for Triangular Norms

The new operator for generating of triangular *t*-norms is described by the expression

$$h_7(x,y) = f^{-1}\left(\frac{f(x)f(y)}{p+f(x)+f(y)}\right), \quad (8)$$

where $x \in [0,1]$, $f(x) \in [0,\infty)$, $f(0)=0$, $\lim_{x\to 1} f(x) = \infty$, $p>0$.

Let us prove that function (8) can be operator for triangular norms generation. In other words, that it satisfies the requirements of a number of axioms, in particular it should be commutative, associative, monotone and satisfy the boundary conditions.

Commutativity: $h_7(x,y) = h_7(y,x)$.

$$h_7(x,y) = f^{-1}\left(\frac{f(x)f(y)}{p+f(x)+f(y)}\right) = h_7(y,x).$$

Associativity: $h_7(h_7(x,y),z) = h_7(x,h_7(y,z))$.

On the one side

$$h_7(h_7(x,y),z) = f^{-1}\left(\frac{f(h_7(x,y))f(z)}{p+f(h_7(x,y))+f(z)}\right) =$$

$$= f^{-1}\left(\frac{\frac{f(x)f(y)}{p+f(x)+f(y)}f(z)}{p+\frac{f(x)f(y)}{p+f(x)+f(y)}f(z)+f(z)}\right) =$$

$$= f^{-1}(f(x)f(y)f(z):$$

$$:(p^2+pf(x)+pf(y)+f(x)f(y)+$$

$$+pf(z)+f(x)f(z)+f(y)f(z))) =$$

$$= f^{-1}(f(x)f(y)f(z):$$

$$:(p(p+f(x)+f(y)+f(z))+$$

$$+f(x)f(y)+f(x)f(z)+f(y)f(z))). \quad (9)$$

On the other side

$$h_7(x,h_7(y,z)) = f^{-1}\left(\frac{f(x)f(h_7(y,z))}{p+f(x)+f(h_7(y,z))}\right) =$$

$$= f^{-1}\left(\frac{f(x)\frac{f(y)f(z)}{p+f(y)+f(z)}}{p+f(x)+\frac{f(y)f(z)}{p+f(y)+f(z)}}\right) =$$

$$= f^{-1}(f(x)f(y)f(z):$$

$$:(p^2+pf(y)+pf(z)+pf(x)+$$

$$+f(x)f(y)+f(x)f(z)+f(y)f(z)) =$$

$$= f^{-1}(f(x)f(y)f(z):$$

$$:(p(p+f(x)+f(y)+f(z))+$$

$$+f(x)f(y)+f(x)f(z)+f(y)f(z))). \quad (10)$$

From comparison (9) and (10) it follows that

$$h_7(h_7(x,y),z) = h_7(x,h_7(y,z))$$

and (8) is an associative function.

Monotonicity: $h_7(x,y) \le h_7(x,z)$ for $y \le z$.

Let $f(x)$ be a monotonically increasing function and such that for $x \in (0,1)$ $f(0)=0$, $\lim_{x\to 1} f(x) = \infty$ and $\{x,y,z\} \in (0,1)$. Suppose that $y \le z$. Then $f(y) \le f(z)$ and

$$\frac{1}{f(y)} \ge \frac{1}{f(z)}. \quad (11)$$

Add $1/f(x)$ to both parts of (11). Then we get

$$\frac{1}{f(x)} + \frac{1}{f(y)} + \ge \frac{1}{f(x)} + \frac{1}{f(z)}. \quad (12)$$

or

$$\frac{f(x)+f(y)}{f(x)f(y)} \ge \frac{f(x)+f(z)}{f(x)f(z)}. \quad (13)$$

On the other side, if $f(y) \le f(z)$ and $f(x) > 0$, then

$$f(x)f(y) \le f(x)f(z)$$

and

$$\frac{1}{f(x)f(y)} \ge \frac{1}{f(x)f(z)}. \quad (14)$$

After multiplying each part of inequality (14) into $p>0$ we have

$$\frac{p}{f(x)f(y)} \ge \frac{p}{f(x)f(z)}. \quad (15)$$

Adding (14) and (15) we get

$$\frac{f(x)+f(y)}{f(x)f(y)} + \frac{p}{f(x)f(y)} \ge \frac{f(x)+f(z)}{f(x)f(z)} + \frac{p}{f(x)f(z)}$$

and

$$\frac{p+f(x)+f(y)}{f(x)f(y)} \ge \frac{p+f(x)+f(z)}{f(x)f(z)},$$

wherefrom

$$\frac{f(x)f(y)}{p+f(x)+f(y)} \le \frac{f(x)f(z)}{p+f(x)+f(z)}$$

since $f^{-1}(\cdot)$ is increasing

$$f^{-1}(\frac{f(x)f(y)}{p+f(x)+f(y)}) \le f^{-1}(\frac{f(x)f(z)}{p+f(x)+f(z)}) .$$

Therefore

$$h_7(x,y) \le h_7(x,z) .$$

Boundary conditions: $h_7(x,1)=x$, $h_7(x,0)=0$.

At $y \to 1$ and $\lim_{y\to 1} f(y)=\infty$, we have

$$\lim_{y\to 1} h_7(x,y) = \lim_{y\to 1} f^{-1}(\frac{f(x)f(y)}{p+f(x)+f(y)}) =$$

$$= \lim_{y\to 1} f^{-1}(\frac{1}{p/(f(x)f(y))+1/f(y)+1/f(x)}) = \lim_{y\to 1} f^{-1}(f(x)) = x$$

$h_7(x,0)=0$. When $y \to 0$

$$\lim_{y\to 0} f^{-1}(\frac{f(x)f(y)}{p+f(x)+f(y)}) =$$

$$= \lim_{y\to 0} f^{-1}(\frac{1}{p/(f(x)f(y))+1/f(y)+1/f(x)}) = \lim_{y\to 0} f^{-1}(0) = 0 .$$

Condition for strict *t*-norm: $h_7(x,x) < x$.

$$h_7(x,x) = f^{-1}(\frac{f(x)f(x)}{p+f(x)+f(x)}) = f^{-1}(f(x)\frac{f(x)}{p+2f(x)}) .$$

Because $f(x)/(p+2f(x)) = k < 1$ for $x \in [0,1]$, then

$$h_7(x,x) = f^{-1}(kf(x)) < x$$

and $h_7(x,y)$ generates Archimedian triangular *t*-norm.

IV. Examples of new triangular norms

Example 1. Let $f(x) = x/(1-x)$, $f^{-1}(x) = x/(1+x)$. Substituting them into (8) we obtain such a new triangular *t*-norm:

$$T(x,y) = \frac{(\frac{x}{1-x}\frac{y}{1-y})/(p+\frac{x}{1-x}+\frac{y}{1-y})}{1+(\frac{x}{1-x}\frac{y}{1-y})/(p+\frac{x}{1-x}+\frac{y}{1-y})} =$$

$$= \frac{xy}{p(1-x)(1-y)+x+y-xy} ,$$

where $p>0$ and the corresponding triangular s-norm

$$S(x,y) = 1 - T(1-x, 1-y) =$$

$$= \frac{pxy + x(1-y) + y(1-x)}{pxy + x(1-y) + y(1-x) + (1-x)(1-y)} .$$

Example 2. Let $f(x) = -\ln(1-x)$, $f^{-1}(x) = 1-\exp(-x)$. Substituting them into (8) we obtain such a new triangular *t*-norm:

$$T(x,y) = 1 - \exp(-\frac{\ln(1-x)\ln(1-y)}{p - \ln(1-x) - \ln(1-y)}) =$$

$$= 1 - \exp(-\frac{\ln(1-x)\ln(1-y)}{p - \ln((1-x)(1-y))})$$

and the corresponding triangular s-norm

$$S(x,y) = 1 - T(1-x, 1-y) =$$

$$= \exp(-\frac{\ln(x)\ln(y)}{p - \ln(xy)}) .$$

The new obtained triangular norms have the parameter p to control their characteristics.

V. Conclusion

A new operator for generating Archimedian triangular norm is presented. Such constructed operator produces parameterized triangular norms. This expands the functional characteristics of logical connectives in fuzzy logic. New triangular norms are more flexible and can be used as logical connectives for fuzzy decision-making systems.

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