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Richards-Klute Equation on Graphs

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Abstract — **This paper contains the formulation of the problem of mass transfer in a porous medium on a graph, on the edges of which the one-dimensional problem of mass transfer is modeled using the Richards-Klute equation. To combine separate Richards-Klute equations into a single system, auxiliary mass balance equations are introduced for the vertices of the graph. Problems of approximation of these equations at the vertices of the graph are also discussed.**

Keywords — mathematical modeling; Richards-Klute equation; numerical methods; graphs.

I. Introduction

The Richards-Clute equation describes the process of mass transfer in a porous medium due to capillarity and gravity. It is widespread in the field of agriculture. It is also used for calculations for the construction of irrigation and drainage systems. The Richards-Klute equation is a quasilinear degenerate elliptic-parabolic partial differential equation. Because of this, very few analytical solutions of this equation are known, and the most common tool for solving the Richards-Klute equation is numerical methods [1]. However, due to the properties of the equation, mentioned before, numerical methods for its solution require a large number of calculations to achieve the required approximation accuracy.

Modifications of standard methods of finite elements [2], finite volumes [3], and finite difference methods [4] are used to increase the efficiency of calculations. Also, papers [5] propose linearization of the Richards-Klute equation. An overview of modern techniques for constructing numerical methods can be found in works [1,5].

II. Problem Formulation

The Richards-Klute equation is an equation of the form

$$\frac{\partial \theta}{\partial t} = \nabla \cdot \left(K \nabla h \right) + \frac{\partial K}{\partial z} + s, \qquad (1)$$

where t is time [s], θ is a water content [-], h is a pressure head [m], K is the hydraulic conductivity [m/s] and s is the source term [s^{-1}]. z is the vertical coordinate, $(x,y,z,t) \in \Omega\times[0,T]$. Form (1) is also known as mixed form of the Richards-Klute equation. Choosing h as primal variable, (1) can be transformed to h- (or head) form

$$C\frac{\partial h}{\partial t} = \nabla \cdot \left(K \nabla h \right) + \frac{\partial K}{\partial z} + s, \qquad (2)$$

where $C=\partial\theta/\partial h$ is the soil moisture capacity [m^{-1}].

Let's consider the three-dimensional Richards-Klute equation, the area Ω of which will be a system of pipes filled with a porous substance (for example, different types of soil). To model the process of mass transfer of a useful substance through such a system of pipes, one can approximate a three-dimensional equation and use numerical methods from works [1-5]. However, to reduce the number of calculations and increase the efficiency of the modeling process, it is advisable to consider some simplification of the system. Let the diameters of the pipes be negligibly small compared to their lengths. In addition, we will assume that the direction and magnitude of the substance flow at a specific point in any pipe does not depend on the position of the point in the cross section of this pipe.

Then, instead of a three-dimensional region Ω, we can consider some graph $G = (V,E)$, the edges $e\in E$ of which will correspond to the pipes in the initial system, and the vertices $v\in V$ will be the connections of these pipes (Fig. 1). Then for each e we have a one-dimensional Richards-Klute equation

$$C_e\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \cdot \left(K_e \left(\frac{\partial h}{\partial x} + g_z \right) \right), x \in [0, L_e], \qquad (3)$$

where C_e and K_e are the characteristics of the porous medium in the pipe corresponding to the edge e, g_z is the gravity impact and its value depends on the direction of the pipe in the gravity field, L_e is length of the pipe. In order to combine the equations on the edges into a single system, we introduce additional binding conditions for the vertices of the graph G. First, we must ensure the continuity of the pressure head h over the entire graph. Then for each vertex we have the following equalities.

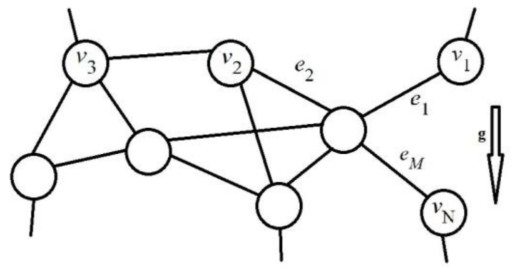

Figure 1. System of pipes presented as graph (g – gravity vector)

$$h(v) = h_e(\delta_e), e \in Adj(v), \quad (4)$$

where δ_e = 0 or L_e depending on which end of e is adjacent to v.

Also, for each vertex, the mass balance condition must be fulfilled, which in terms of the Richards-Klute equation has the following form.

$$\frac{\partial \theta(v)}{\partial t} = \sum_{e \in Adj(v)} q_e(\delta_e), \quad (5)$$

where q_e is a positive flux through the boundary δ_e of the edge e.

Adding initial conditions for edges and boundary conditions for some vertices to (3-5), we obtain a system of equations for modeling the mass transfer process on the graph G.

III. Numerical Approximation

Since the parameters of the medium on different edges of the graph may differ, it is necessary to provide a rule for the discretization of the mass balance equation (5) for further application of numerical methods. This equation is similar to the condition at the boundary between different layers of the non-homogeneous medium for the one-dimensional Richards-Klute equation. Since equation (4) allows us to identify the vertex node and the end nodes of the edges adjacent to this vertex during discretization, we have the following mass balance equation.

$$\begin{aligned}&\sum_{i=1}^{m_N} C_i^{N_i-\frac{1}{2}} \frac{h_v^{j+1}-h_v^{j}}{\Delta t} \\ &+\sum_{i=1}^{m_0} C_i^{\frac{1}{2}} \frac{h_v^{j+1}-h_v^{j}}{\Delta t} \\ &=\sum_{i=1}^{m_N} \frac{1}{\Delta x_i} K_i^{N_i-\frac{1}{2}} \left(\frac{h_{i,N_i-1}^{j+1}-h_v^{j+1}}{\Delta x_i}+g_{i,z}\right) \\ &-\sum_{i=1}^{m_0} \frac{1}{\Delta x_i} K_i^{\frac{1}{2}} \left(\frac{h_{i,1}^{j+1}-h_v^{j+1}}{\Delta x_i}+g_{i,z}\right),\end{aligned} \quad (6)$$

where m_0 is number of edges, for which δ_e = 0, m_N is number of edges, for which $\delta_e = L_e$, j is current and j+1 is the next time step, Δx_i, $g_{i,z}$, K_i, C_i are the spatial step, gravity impact, hydraulic conductivity and soil moisture capacity for the edge e_i respectively. K_i and C_i are calculated at the corresponding point of edge e_i (see Fig. 2).

Let's consider an example of Richards-Klute equation and its numerical solution obtained using the given discretization. Let graph G looks like on the Fig. 3. For every edge e of G we consider the one-dimensional Richards-Klute equation (3) with the next parameters.

$$\theta(h) = \frac{\alpha(\theta_s - \theta_r)}{\alpha + |h|^{\beta}} + \theta_r, K(h) = K_s \frac{A}{A + |h|^{\gamma}}, \quad (7)$$

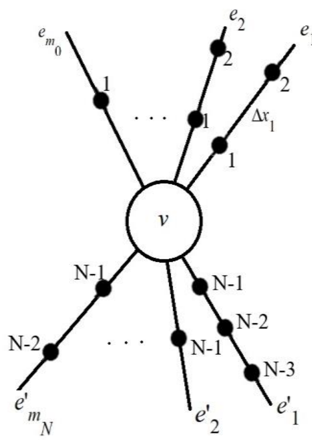

Figure 2. Node v and edges adjacent to it

where α = 1.611×106, θ_r = 0.075, θ_s = 0.287, γ = 4.74, A = 1.175×106, K_s = 0.00944, L = 40, g_z = 1 for every edge, β_1 = 4.96, β_2 = 2.96, β_3 = 3.96. Initial condition is h = −41.5 for every edge and vertex. Boundary conditions on the vertices are h_1 = −41.5, h_2 = −20.7, h_3 = −20.7. Parameters of discretization are Δt = 1, Δz = 0.8 for all edges. Fig. 4 contains graphs of approximate solution at the time T = 200.

Due to the different parameters of the media in the edges, the substance spread much faster through edge e_3 and began to spread up edge e_2 through the vertex v_4. Note that the matrix of the system of linear algebraic equations, which is formed to find an approximate solution in the next step, is not tridiagonal in this case. However, in the general case, the matrix is symmetric, sparse, and has the property of diagonal dominanceю It allows the use of effective numerical methods and increases the efficiency of the calculation process compared to the standard three-dimensional approximation of the Richards-Klute equation.

IV. Conclusions

A mass transfer process in porous media on graphs model was built using the Richards-Klute equation. This model serves as a replacement for the three-dimensional classical Richards-Klute equation to increase the efficiency of an approximate solution finding process. Mass balance equations and their numerical approximation for graph vertices were discussed. Numerical experiments were carried out.

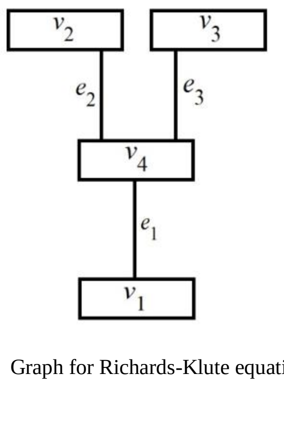

Figure 3. Graph for Richards-Klute equation (3), (7)

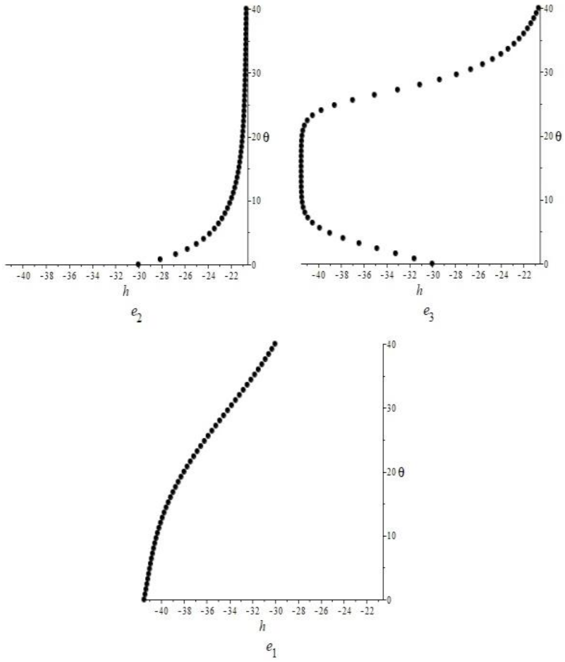

Figure 4. Solution of Richards-Klute equation on graph

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