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Methods for simplifying mathematical models of dynamical systems

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Abstract—**We consider the problem of simplifying a system of ordinary differential equations obtained as a result of reconstruction from a single observed variable. To solve the problem, we propose two methods: analytical and numerical-analytical. Both methods were applied to a third-order system of ordinary differential equations with polynomial right-hand sides and the results of the methods were compared.**

Keywords—dynamical system; time series; model simplification; analytical method; numerical-analytical method.

I. Introduction

The problem of identifying a system of ordinary differential equations (ODE) from time series of observed variables [1] is often occurred in nonlinear dynamics. One of the special cases of this problem is the reconstruction problem, i.e. identification of an ODE system based on one observed variable. If only numerical methods are used for reconstruction (for example, the Bock's algorithm [2]), then the result can be an ODE system containing redundant terms that have no physical meaning. In such a situation, the task of simplifying the ODE system may arise. Methods for solving this problem are discussed in this paper.

II. Formulation of the Problem

We assume that the ODE system reconstructed by the numerical method has the form

$$\begin{cases} \dot{x}_1 = a_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_1^2 + a_5x_1x_2 \\ \qquad + a_6x_1x_3 + a_7x_2^2 + a_8x_2x_3 + a_9x_3^2, \\ \dot{x}_2 = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_1^2 + b_5x_1x_2 \\ \qquad + b_6x_1x_3 + b_7x_2^2 + b_8x_2x_3 + b_9x_3^2, \\ \dot{x}_3 = c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1^2 + c_5x_1x_2 \\ \qquad + c_6x_1x_3 + c_7x_2^2 + c_8x_2x_3 + c_9x_3^2, \end{cases} \tag{1}$$

where a_i, b_i, c_i, $i = 0, \ldots, 9$, are constants, x_1 – observed variable. According to [3], a system that precisely describes the dynamics of a studied process will be called the original system (OS). We will assume that the general form of the OS equations and the numerical values of the coefficients were obtained using one of the well-known numerical methods and some of the coefficients of the equations are redundant. Even if some of the found coefficients of OS (1) will have values close to zero, this would not mean that these coefficients can be neglected, since it could sometimes lead to a significant error. Since the structure of the system (1) may be not optimal due to containing redundant terms, we will call it unoptimized original system (UOS). One needs to find instead of UOS another system that we will call a minimized original system (MOS). It has the form

$$\begin{cases} \dot{x}_1 = P_1(x_1, x_2, x_3), \\ \dot{x}_2 = P_2(x_1, x_2, x_3), \\ \dot{x}_3 = P_3(x_1, x_2, x_3), \end{cases} \tag{2}$$

where P_1, P_2, P_3 are polynomials, like in system (1), but the total number N_{MOS} of nonzero coefficients in the right-hand sides of system (2) is subject to the condition $N_{MOS} < N_{UOS}$, where N_{UOS} is the total number of nonzero coefficients in the UOS. In the process of simplifying the UOS equations, the auxiliary type of ODE systems proposed in [3, 4] will be used. We will call a differential model (DM) [1] a system of ODEs of the form

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \dot{y}_3 = F(y_1, y_2, y_3), \end{cases} \tag{3}$$

where F is a polynomial function or a ratio of polynomials. In this case, the observed variable DM y_1 coincides with the observed variable OS, and all DM coefficients can be analytically expressed through OS coefficients. This fact allows us to formulate an obvious statement used in [5, 6]: two different original systems, which have the same observable, have the same DM that was obtained from this observable. This statement permits to vary coefficients of the OS in such a way that the coefficients of the DM would remain unchanged. Taking into account the above, two methods can be proposed to simplify the UOS structure – the analytical method [7] and the numerical-analytical one [8].

III. The analytical method

The analytical method for simplifying the UOS structure [7] uses analytical relationships between the OS and DM coefficients as follows. To simplify the structure of the UOS, it is sufficient to make some of its coefficients equal to zero and to change values of the remaining coefficients in such a way that the DM that corresponds to the MOS would coincide with the DM corresponding to the UOS. To do this, all UOS coefficients can be divided into three sets – uniquely determined, preset and adjustable. Uniquely determined coefficients can be analytically expressed in terms of DM coefficients, so they cannot be excluded from the UOS equations. Adjustable coefficients can be expressed in terms of DM coefficients, uniquely determined and predefined UOS coefficients. Therefore, when changing the values of the preset coefficients, the adjustable coefficients will change so that the DM coefficients remain unchanged. Thus, simplifying the UOS equations can be done in the following sequence:

1) To get analytic relations between coefficients of the DM and coefficients of the UOS.

2) Separate the set of uniquely determined coefficients of the UOS using analytic relations.

3) Make an analysis of relations for coefficients for the DM and obtain relations that define adjustable coefficients of the UOS in terms of coefficients of the DM, uniquely determined coefficients and preset coefficients of the UOS.

4) Obtain values of the coefficients of the MOS by substituting, into the relations obtained in Step 3, values of the coefficients of the DM, the uniquely determined coefficients of the UOS, and zeros for the preset coefficients.

5) Find the initial values of the variables in MOS so that its observable coincides with the observable in the UOS.

Let us remark that choice of different sets of preset coefficients could lead to MOSs with different structures.

The analytical method was applied to simplify system (39) from [9] that was obtained from Lorenz system [10] using the “Ansatz library” method. The system contains N_{UOS} = 21 coefficients, instead of 7 coefficients as in the Lorenz system. This Lorenz-like system has the form

$$\begin{cases} \dot{x}_1 = a_0 + a_1 x_1 + a_2 x_2 + a_4 x_1^2, \\ \dot{x}_2 = b_0 + b_1 x_1 + b_2 x_2 + b_4 x_1^2 + b_5 x_1 x_2 \\ \qquad + b_6 x_1 x_3 + b_7 x_2^2, \\ \dot{x}_3 = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1^2 + c_5 x_1 x_2 \\ \qquad + c_6 x_1 x_3 + c_7 x_2^2 + c_8 x_2 x_3 + c_9 x_3^2. \end{cases} \tag{4}$$

The variable x_1 was taken as an observable. A DM that corresponds to OS has the form of a fractional rational function with 35 coefficients in the numerator and 1 coefficient in the denominator. The values of the DM coefficients were analytically calculated based on the values of the UOS coefficients. The analytical method outlined above was then applied. It was found that in system (4) there is only one uniquely determined coefficient – a_2. Coefficients a_0, b_1, b_2, b_4, c_8 were taken as preset coefficients. Resetting these coefficients to zero allowed us to obtain MOS with $N_{MOS} = 16$ coefficients. Wherein we set to zero the preset coefficients with different values, for example, $b_2 = 11.617$. Hence, this approach permits to set to zero not only small coefficients of the UOS.

A comparison of the time series x_1 for the UOS and the MOS has been carried out. For this purpose, the initial conditions UOS and MOS were chosen at which their observed variables coincide and both systems were integrated over an interval of 50 s with a sampling step of 0.001 s. To estimate the degree of coincidence, we calculate the value of coincidence time, t_c, for which the time series for $x_1(t)$ and $x'_1(t)$ in the UOS and the MOS, correspondingly, differ by no more than Δ,

$$t_c = \max\left\{ t : \forall \tau < t, \left| x_1(\tau) - x_1'(\tau) \right| < \Delta \right\}. \tag{5}$$

The Δ value was 10% of the range of the time series in the UOS (4). With such a Δ, we have obtained $t_c = 32.916$ s. The discrepancy between the time series may be explained by chaotic character of the time series and numerical errors.

To make a comparison, we have considered the structure optimization procedure for UOS (4) in the case of periodic oscillations. To get periodic oscillations, we changed value of a_1 with other coefficients of the UOS remaining unchanged. As a result of simplifying UOS, MOS was obtained with the same structure as in the previous example, but with different coefficient values. Integrating this MOS resulted in t_c = 50 s, that is, the time series coincided with acceptable precision on the entire integration interval. This example confirms that the main reason for the deviation of the UOS and the MOS time series is the chaotic nature of the oscillations.

As we have mentioned before and that can be seen from the given procedure, it uses no numerical methods. No numerical methods are also used when getting to the OS from the DM. This fact makes the proposed approach different from the Ansatz library method, where one uses a numerical procedure “to invert the maps between the differential models and corresponding ansatz models” [11].

IV. The numerical-analytical method

The numerical-analytical method for simplifying the structure of UOS [8] is based on the idea that MOS can be obtained if we assume an approximate, rather than exact, coincidence of the time series of observed variables UOS and DM. In this case, the DM may have a simpler structure and can correspond to a MOS with fewer terms in the equations. To obtain an approximate DM, you can use the numerical method [4], which allows you to obtain numerical values of the DM coefficients from the time series of the observed variable UOS. As a measure of the coincidence of the time series $x_1(t)$ and $y_1(t)$ UOS and DM, respectively, the relative root mean square was used:

$$\delta = \frac{\sqrt{\frac{1}{m}\sum_{j=0}^{m-1}\left(y_1(j\Delta t)-x_1(j\Delta t)\right)^2}}{\sqrt{\frac{1}{m}\sum_{j=0}^{m-1}\left(x_1(j\Delta t)\right)^2}}, \tag{6}$$

where m is the number of time series points; Δt is the sampling step of the time series.

The detailed implementation of the numerical-analytical method is as follows:

1) According to the time series of the observed UOS variable, perform a numerical reconstruction of the DM. During numerical reconstruction, in addition to the values of the DM coefficients, their significance [11] is also calculated which is determined by the formula

$$\alpha_k = \frac{\left|M(N_k)\right|}{\sigma(N_k)}, \tag{7}$$

where $M(N_k)$ and $\sigma(N_k)$ are the mean value and root mean square of the DM coefficient N_k, respectively. The significance value is used to identify which coefficients are present (or absent) in the DM equations. Higher values of α_k correspond to the coefficients that are present in DM.

2) Simplify the DM structure using the significance values of its coefficients.

3) If the DM structure obtained in Step 2 by the numerical method does not correspond to the DM structure obtained analytically, then the structure from Step 2 must be made corresponding to the analytical one. Namely, it is necessary to add or remove a coefficient from DM, regardless of its significance, calculated in Step 2, otherwise the analytical transition from OS to DM will become impossible. That is, it will be impossible to obtain a relations connecting the DM and the OS coefficients.

4) Reduce the number of DM and OS coefficients by repeating Steps 1-3 until δ (6) is within acceptable limits.

5) Perform an analytical transition from a simplified DM to an OS that can be used as a MOS.

The numerical-analytical method was applied to UOS (4) To obtain a time series of the observed variable $x_1(t)$, the system was integrated over an interval of 20 s with a sampling step of 0.002 s. Using the time series, a numerical method was used to obtain a DM with 36 coefficients, for which $\delta = 0.29\%$. Simplification of the DM in accordance with Steps 1-3 of the proposed sequence of actions made it possible to consistently obtain a DM with 15 coefficients ($\delta = 0.77\%$), 11 coefficients ($\delta = 2.82\%$) and 12 coefficients ($\delta = 2.45\%$). The last DM corresponds to MOS with $N_{MOS} = 9$ coefficients – a_2, b_1, b_2, b_6, b_7, c_3, c_4, c_5, c_7. When integrating MOS, a time series of its observable $x'_1(t)$ was obtained, for which the value of coincidence time (5) was $t_c = 3.7$ s. After specifying the initial MOS conditions, the time series $x'_1(t)$ was obtained, for which $t_c = 20$ s.

V. Comparison of methods for simplifying the UOS structure

The considered methods are intended to simplify the structure of the equations of the ODE system obtained by reconstruction from the time series of one observed variable. Unlike most reconstruction methods, the considered methods use information about the equations and values of the UOS coefficients, rather than time series, as input data. This allows the researcher to simplify the UOS, which has been obtained in the past by other researchers and for which time series are not available. To apply both methods, the following conditions must be satisfied: an ODE system contains (presumably) excess terms and it is possible to pass analytically to a DM from the OS.

The analytical method allows to researcher to obtain a MOS for which the observed variable is proven to coincide with the observed variable UOS. Therefore, using the analytical method, it is possible to exclude even coefficients with large absolute values from the UOS equations without loss of precision, which can be important in the case of chaotic systems. Another advantage of the analytical method is that, in general, several MOS structures can be obtained. Then it is possible to choose the one from them, based, for example, on the physical meaning of the equations.

The numerical-analytical method allows to researcher to obtain MOS with fewer coefficients than in the case of using the analytical method. In this case, MOS only approximately reproduces the observed UOS time series. The numerical-analytical method is intended for cases where it is acceptable for MOS to correctly reproduce a time series over a shorter time interval than UOS.

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