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Optimal Control of the Position of Moisture Transfer Sources in Porous Media

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Abstract — **The paper proposes an approach for finding the optimal position of sources of known power for the quasi-linear Richards equation in a rectangular area. The Kirchhoff transformation is applied with the introduced scaling of coordinates and powers of submerged sources, which allows formulating a dimensionless problem. The task of this study is to find the position of submerged sources - such that the distribution of moisture at the final moment of time is close to the given values or the given target function.**

Keywords— Richards equation; control; optimization; porous media; moisture transfer.

I. Introduction

The Richards equation in its original nonlinear form is discussed for quite a long time due to its complexity. A review of achievements and problems related to its solution can be seen for example, in M. W. Farthing [1], Y. Zha [2]. In the case of variably saturated flow in inhomogeneous porous media with layers of different properties, H. Suk [3] proposed a numerical solution method that is applied after the Kirchhoff transformation.

Due to the presence of dynamic capillarity in the system, in C. J. Van Duijn [4] proposed an extension of the Richards equation that includes non-equilibrium effects, analysis of water pressure and saturation. The parameters describing the pore structure were obtained by L. J. Cooper [5] using simulation based on three-dimensional computer tomography images for a soil sample.

K. Kumar [6] provides a catalog of effective models that have been confirmed numerical calculations to describe the flow in an unsaturated porous medium containing a crack. Nonlinear solver of the Richards equation with the help of variable substitution, in particular, the introduction of a dummy variable, proposed by S. Bassetto [7].

Several methods of implicit and semi-implicit time discretization were investigated in S. Keita [8] with second-order accuracy, formulas were used extrapolations and approximation by Taylor series for time discretization nonlinear members.

Recently, the Richards equation is usually solved using the help of local methods, for example, the method of finite differences, finite elements, and finite volumes [9] with the application of iterative methods of the Newton type.

Therefore, the study of the Richards equation continues even now, as evidenced above specified scientific results for the last 5 years.

II. Two-dimensional Modeling Problem

A. Equation Describing the Process

Consider the mathematical model of moisture transfer for the soil area $\Omega=\{(x,y):0<x<l_1, 0<y<l_2\}$, where l_1 and l_2 are the width and depth, respectively, with the initial moisture equal to , the fixed moisture at the boundary, and the determined target moisture distribution at the final moment of time $\varphi(x,y)$. We consider the fluid to be incompressible, and the pressure on the system is constant.

The model is represented by the Richards equation with boundary conditions of the first type:

$$\frac{\partial\omega}{\partial t}-\frac{\partial}{\partial x}\left[K_x(\omega)\frac{\partial H}{\partial x}\right]-\frac{\partial}{\partial y}\left[K_y(\omega)\frac{\partial H}{\partial y}\right]=$$

$$=\sum_{m=1}^{M} Q_m F(x-x_m, y-y_m)$$

$$(x,y)\in\Omega, t\in(0,T]; \quad (1)$$

$$\omega|_{\partial\Omega}=\omega_0,\ \omega|_{t=0}=\omega_0. \quad (2)$$

Here ω — humidity, ω_0 — permanent humidity, ψ — pressure height, $H=\psi(\omega)-y$ — hydrodynamic head, $K_x(\omega)$ — water permeability along the axis Ox, $K_y(\omega)$ — water permeability along the axis Oy, $F(x-x_m, y-y_m)\in L_2((0,T)\times\Omega)$ — a function determining the influence on the system of the source located at the point (x_m, y_m). The diffusivity function along the Oy axis is $D_y(\omega) = K_y(\omega)\frac{d\psi}{d\omega} = e^{0.5\omega}$.

The right-hand side of (1) contains a set of sources of known power Q_m, where $0 \le Q_m \le Q_{\max}, m = 1,...,M$. The problem is considered only for the unsaturated case.

B. Kirchhoff Transformation and Scaling

Let us assume that the water permeability along the axes is presented in the form $K_x(\omega) = k_1 \cdot k(\omega)$, $K_y(\omega) = k_2 \cdot k(\omega)$ where k_1, k_2, are coefficients of wet conductivity along the axes Ox, Oy and $k(\omega)$ — moisture conductivity. Without limiting in general, let's put $k_1 = k_2$. Suppose that:

$$\frac{1}{D_y(\omega)} \cdot \frac{dK_y(\omega)}{d\omega} = \ell = const,$$

$$\frac{\partial \omega}{\partial t} = \frac{Q^* k_2 \beta_2}{4\pi k_1} \frac{1}{D_y(\omega)} \frac{\partial \Theta}{\partial t} \cong \frac{k_2 \beta_2^3 Q^*}{4\pi k_1} \frac{\partial \Theta}{\partial \tau},$$

where $\beta_2 = 0.5\ell$, $\beta_1 = \sqrt{\frac{k_2}{k_1}}\beta_2$, $\alpha = \frac{\langle D_y \rangle \beta_2^2}{T}$, $\xi = \frac{\beta_1}{l_1} x$, $\zeta = \frac{\beta_2}{l_2} y$, $\tau = \alpha t$, $\langle D_y \rangle$ — average value of $D_y(\omega)$ in the given area.

Let's apply the Kirchhoff transformation:

$$\Theta(\omega) = \frac{4\pi k_1}{Q^* k_2 \beta_2} \int_{\omega_0}^{\omega} D_y(\omega) d\omega, \qquad (3)$$

where Q^* is the source power scaling factor.

Let's introduce the appropriate notation for the power of the sources after scaling: $q_j = \frac{Q_j}{Q^*}$, $\Omega_0 = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ — the scaled area Ω, $\partial\Omega_0$ — its boundary. Next, we will consider the area $U = [0,1] \times \Omega_0$. The equation obtained as a result of the transformation will be considered relative to Θ.

Applying the Kirchhoff transformation (3), for the initial and boundary conditions (2), we obtain the equality of the upper and lower limits of integration and $\frac{4\pi k_1}{Q^* k_2 \beta_2} \int_{\omega_0}^{\omega_0} D_y(\omega) d\omega = 0$. Inside the domain, the transformed equation (1) has the form:

$$L(\Theta) \equiv \frac{\partial \Theta}{\partial \tau} - \frac{\partial^2 \Theta}{\partial \xi^2} - \frac{\partial^2 \Theta}{\partial \zeta^2} + 2\frac{\partial \Theta}{\partial \zeta} =$$

$$= \sum_{m=1}^{M} q_m \cdot F(\xi - \xi_m, \zeta - \zeta_m)$$

$$(\xi, \zeta) \in \Omega_0, \tau \in (0,1], \qquad (4)$$

$$\Theta|_{\partial\Omega_0} = 0, \ \Theta|_{\tau=0} = 0. \qquad (5)$$

III. OPTIMIZATION PROBLEM

To formulate the optimization problem, consider the averaging of humidity around the points. We denote by r_m, $m = 1,...,M$ the coordinates of the centers of the sources having the corresponding powers q_m. We define the target values of humidity $\varphi(1, \xi_s, \zeta_s)$ as an average for the humidity $\Theta(1, \xi, \zeta)$ around the selected points $(\xi_s, \zeta_s) \in \Omega_0$, $s = 1,...,S$. The aim of the study is to find the values of r_m, $m = 1,...,M$, that minimize the value of the square of the difference between $\Theta(1, \xi_s, \zeta_s)$ (the solution of the direct problem (4) by the selected coordinates of the location of the sources) and $\varphi(1, \xi_s, \zeta_s)$. Accordingly, the optimal control is a M-dimensional vector of pairs of source coordinates from the set $([0,1] \times [0,1])^M$. We will consider the objective function in the form:

$$J(\bar{\xi}, \bar{\zeta}) = \int_U \left(\left(\Theta(1, \xi, \zeta) - \varphi(1, \xi, \zeta) \right)^2 \right) dU \qquad (6)$$

where $\bar{\xi} = (\xi_1, ..., \zeta_M)^T$, $\bar{\zeta} = (\zeta_1, ..., \zeta_M)^T$ — vectors of source coordinates, with the help of which control is carried out. Therefore, the optimal choice of power sources is reduced to the minimization of the functional

$$J(\overline{\xi^*}, \overline{\zeta^*}) = \min_{\bar{\xi}, \bar{\zeta} \in [0,1]^M} J(\bar{\xi}, \bar{\zeta}). \qquad (7)$$

IV. MATHEMATICAL MODELING

A. Numerical Representation

After the Kirchhoff transformation to linearize the operator, we divide it into time intervals and introduce a grid with a uniform step in spatial coordinates.

To choose the step of division by time and spatial coordinates, we use the condition of stability of the explicit scheme in the form of $\tilde{\tau} \le \frac{h^2}{2}$. Calculations were performed with the step $h = \frac{1}{20}$ partitioning by space and $\tilde{\tau} = 10^{-4}$ for partitioning the time interval. Next, we denote the values Θ^n, φ^n as averaged over the spatial region around the grid node at the moment of time $\tilde{\tau} \cdot n$.

Next step is replacing the derivatives in space and time with their differential approximation (we present an explicit scheme). Based on the known current approximation of the location of each source $(\xi_m, \zeta_m) \in \Omega_0$, $m = 1,...,M$, find the distribution of moisture $\Theta_{i,j}^n = \Theta\left(\frac{i}{20}, \frac{j}{20}, n \cdot 10^{-4}\right)$, $i, j = 0,...,20$, at each time step $n = 1,...,N$, based on the known distribution of moisture on the boundaries and at the initial moment of time:

$$\frac{\Theta_{ik}^{n+1}-\Theta_{ik}^{n}}{\tilde{\tau}}=\frac{\Theta_{i+1k}^{n}-2\Theta_{ik}^{n}+\Theta_{i-1k}^{n}}{h^2}+\frac{\Theta_{ik+1}^{n}-2\Theta_{ik}^{n}+\Theta_{ik-1}^{n}}{h^2}-$$
$$-2\frac{\Theta_{ik+1}^{n}-\Theta_{ik-1}^{n}}{2h}+\sum_{m=1}^{M}F(\xi_i,\zeta_k)\cdot q_m;\ \Theta_{ik}^{0}=0.$$

B. Idea used for optimization

Find maximal value locations of desired function.

a) Perform analysis, where extremums are located and divide the area into sub-areas with 1 maximal value on each.

b) Set a source into every maximum's location from highest power to lowest till all sources are placed.

2) Adding extra limitations and testing.

a) Sub-area selection: Perform a simulation with 1 source of lowest power. All locations, where target function has much lower values than humidity at source position are excluded from search.

b) Initial testing: Set all sources into points with maximal values of target function to get humidity simulation results. Also perform testing with symmetric placement of 2 or several sources around the maximal value point.

c) Further testing: Change position of 1 source by 1 horizontal, vertical or diagonal position compared to previous tests. Compare results with achieved previously using quality functional. Then change position of another source, replacement of which along same direction did not lead to worse result in the past. Finish testing when required accuracy is achieved, or all directions of movement for sources make result worse.

C. Simulation results

For simplicity of explanation, we will demonstrate the performance of the idea for two sources with equal power using the proposed method. To simulate moistening with the help of sources, we introduce the function corresponding to the source located at the point (0.5, 0.5):

$$F(\xi,\zeta)=\begin{cases}\dfrac{750\cdot\sqrt{\left|\frac{1}{100}-(\xi-0.5)^2-(\zeta-0.5)^2\right|}}{2\pi}, & \sqrt{(\xi-0.5)^2+(\zeta-0.5)^2}\le 0.1,\\ 0 \text{ else;}\end{cases}$$

It is easy to check that this function is bounded and belongs to the space $L_2(U)$, and is also continuous. To locate sources with other coordinates, we will use a substitution with a shifted argument. Desired function is chosen as:

$$\varphi^n=\varphi(1,\xi,\zeta)=\begin{cases}0.5\cdot\sqrt{\left|\frac{4}{100}-(\xi-0.5)^2-(\zeta-0.5)^2\right|}, & \sqrt{(\xi-0.5)^2+(\zeta-0.5)^2}\le 0.2,\\ 0 \text{ else;}\end{cases}$$

Let's calculate humidity distribution for the case of one source with power 0.5 and determine the maximum value obtained in the grid nodes. Then add one source with same power and test different source positions to find an acceptable solution. Desired function differs from possible results, so we will search for average modular difference of values between Θ^n reached and φ^n.

TABLE I. HUMIDITY DISTRIBUTION AND POSITION TESTING

Source location	***Average on all*** $\left\|\Theta^n-\varphi^n\right\|$	***Average on central part*** $\left\|\Theta^n-\varphi^n\right\|$
(0.45, 0.45), (0.55, 0.55)	0.01117	0.02456
(0.45, 0.45), (0.55, 0.50)	0.01111	0.02434
(0.45, 0.5), (0.55, 0.5)	0.01105	0.02411
(0.45, 0.5), (0.5, 0.5)	0.01098	0.02376
(0.4, 0.5), (0.5, 0.5)	0.01117	0.02434
(0.45, 0.5), (0.6, 0.5)	0.01124	0.02492
(0.4, 0.5), (0.6, 0.5)	0.01142	0.02545

According to results, solution (0.45, 0.5), (0.5, 0.5) represents the highest found accuracy and testing finished much faster than in case of checking all possible positions. The desired function has different geometry compared to modeling results.

The difference between projections on Oy axis for desired and achieved humidity distribution is significant, still the proposed idea worked good enough and optimal source position was found. As the desired function has symmetry, the solution is not unique but effective.

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