Equilibrium state as a compromise in the struggle of opponents

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Abstract—**Mathematical models describing the conflict interaction between alternative opponents are studied. It is assumed that the adversaries are indestructible, located in different regions of the resource space, and receive external support in their struggle with each other. The main questions concern the compromise states of equilibrium (a certain type of fixed points) of the associated dynamic conflict system. Namely, the existence of such states, their stability, and the dominant side in each region. It has been established that states of equilibrium compromise arise only in the presence of external influences (supports) necessarily for both opponents and only some of them are stable with non-trivial basins of attraction. It was also found that with insufficient external support, the dominant opponent in each of the regions can sharply lose its position.**

Keywords—dynamical system of conflict, law of conflict interaction, stochastic vector, limit state, fixed point, equilibrium, compromise, stability, discrete measure, dissemination of beliefs.

We study the mathematical models of dynamical system describing the conflict interaction between alternative opponents. It is assumed that opponents are not destroyed and receive various external supports in different regions Ω_i of their life space

$$\Omega = \bigcup_{i=1}^{\infty} \Omega_i, \qquad 2 \le m < \infty.$$

Under different external influences (supports) for both opponents we establish the existence of equilibrium (compromise states) for such system.

The law of conflict interaction between a couple of alternative opponents, say A and B , has the view

$$\frac{d}{dt}P_i^A = \lambda P_i^A(1 - P_i^B),$$
$$\frac{d}{dt}P_i^B = \lambda P_i^B(1 - P_i^A), \quad i \in \overline{1,m}, \qquad (1)$$

where $P_i^A \equiv P^A(\Omega_i, t)$ ($P_i^B \equiv P^B(\Omega_i, t)$) denotes an independent probability presence of opponent A (B) is the region Ω_i at time t (λ stands for the probability normalization). Law (1) has the following heuristic interpretation: which of the adversaries A or B should be or not be in each of the regions Ω_i at time t after each act of the battle?

Actually, here we use discrete time and therefore the above system *2m* of differential equations (1) are transformed into a system of difference equations through the coordinates of stochastic vectors $\mathbf{p}^t = (p_1^t, \dots, p_m^t), \mathbf{r}^t = (r_1^t, \dots, r_m^t)$ from the space $\mathbb{R}^m_{+,1}$:

$$p_i^{t+1} = \lambda p_i^t(1 - r_i^t),$$
$$r_i^{t+1} = \lambda r_i^t(1 - p_i^t), \quad t=0,1,\dots, \qquad (2)$$

where $p_i^t \equiv P^A(\Omega_i, t)$, $r_i^t \equiv P^B(\Omega_i, t)$. Due to normalization, we have:

$$\sum_{i=1}^{m} p_i^t = 1 = \sum_{i=1}^{m} r_i^t.$$

It is worth noting that dynamical systems of conflict in the form

$$\{\mathbf{p}^t, \mathbf{r}^t\} \to \{\mathbf{p}^{t+1}, \mathbf{r}^{t+1}\}, \quad t = 0,1, \dots$$

have already been investigated in a number of publications (see [1-8] and the references given there). The basic result confirms the convergence of trajectories of such systems to equilibrium states. This result is known as the Conflict Theorem. It can be formulated shortly as follows.

Each trajectory $\{\mathbf{p}^t, \mathbf{r}^t\}$ of the dynamical system of conflict generated by the system of equations (2) with an arbitrary starting point $\{\mathbf{p}^0, \mathbf{r}^0\}$ given by a pair of stochastic vectors $\mathbf{p}^0, \mathbf{r}^0 \in \mathbb{R}^m_{+,1}$ such that $(\mathbf{p}^0, \mathbf{r}^0) \neq 1$, converges to the limit state (fixed point),

$$\{\mathbf{p}^t, \mathbf{r}^t\} \to \{\mathbf{p}^\infty, \mathbf{r}^\infty\}, \quad t \to \infty,$$

which consists of two orthogonal vectors, $\mathbf{p}^\infty \perp \mathbf{r}^\infty$.

Thus, only one of the opponents A or B can be remained in each region. That is, one of the two is always fulfilled, $p_i^\infty \geq 0, r_i^\infty = 0$, or $p_i^\infty = 0, r_i^\infty \geq 0$. There is no compromise in any of areas. Therefore, it is impossible that both limiting coordinates values $p_i^\infty > 0, r_i^\infty > 0$ are non-zero.

The question of the existence of balanced compromise states, when the struggle of the enemies ends not with victory or defeat, but with the constant presence of both opponents in at least one disputed region, has a positive answer only if the opponents receive (external) help (reinforcement). Mathematically, this is written by entering coordinates shifts in the vectors of the probability of the presence of opponents in different regions:

$$p_i^t + h_{pi}, \quad r_i^t + h_{ri}.$$

Such shifts which produced by the external resource generate some kind of attraction for both of opponents. We remark that a series of models with pure attractive interaction have already been investigated in the works [9-11]. Here we show that the above attraction able to lead to compromise equilibrium. Our new result establishes the existence of compromise states (simultaneous presence of opponents in a fixed region) under receiving of external help for both ones.

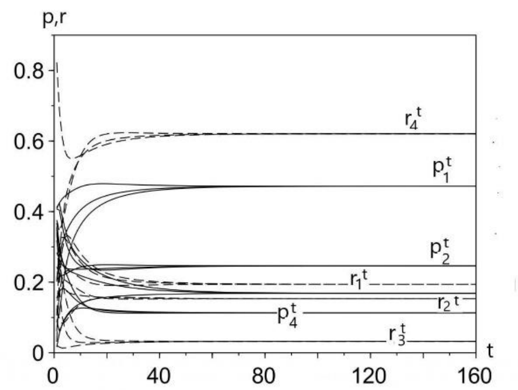

Figure 1. An example of a limit compromise equilibrium when coordinatesshifts arenon-zero in all regions

The dynamical equations with external interference have the form

$$p_i^{t+1} = \frac{(p_i^t + h_{pi} r_i^t)(1 - r_i^t)}{z_p^t},$$
$$r_i^{t+1} = \frac{(r_i^t + h_{ri} p_i^t)(1 - p_i^t)}{z_r^t}, \quad i \in \overline{1, m}. \tag{3}$$

where parameters $h_{pi}, h_{ri} \geq 0$ are interpreted as help (reinforcement, influence) for opponents *A*, and *B*, respectively, in the region Ω_i.

We prove that for dynamical conflict systems given by equations (3), there exist equilibrium compromise states. That is, the compromise values p, r in regions Ω_i are determined only by the parameters of external influence. We illustrate this fact in Figure1.

Finally, we note that the obtained results are suitable for application in the theory of formation and dissemination of beliefs and for finding conditions for establishing consensus between different opinions in the sense of the works [12, 13].

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