On the Rod Heating Problem by Sources on the Class of Zonal Controls Using the Current and Past State at Measurement Points

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Abstract—**We study an optimal feedback control problem for the rod heating process by means of lumped sources. The control actions are the powers of the sources, the values of which are defined on the class of zonal controls. The values of the parameters of zonal control actions are determined by subsets of the state space, to which belong the values of the process state at the measurement points at the current and past time moments. The posed problem is reduced to a parametric optimal control problem on determining a finite-dimensional vector of values of the parameters of zonal control actions. We have obtained optimality conditions for the values of the parameters of zonal control actions. These conditions contain formulas for the gradient of the objective functional with respect to the optimizable parameters. They make it possible to solve the reduced problem numerically with the application of efficient first-order optimization methods.**

Keywords—feedback control; zonal control; distributed parameters system; heat conduction process; gradient of functional.

I. Introduction

It is known that one of the important areas in the modern automatic control theory is the theory of control of systems with distributed parameters. The synthesis problems for distributed control systems are, in most cases, more complex than lumped systems due to the characteristics of distributed objects. Distributed control objects include, for example, chemical-technological, radiation, aerodynamic, and hydrodynamic processes, heat conduction and diffusion processes, processes associated with the movement of elastic structures, etc. The absence of a formalized methodological approach for solving problems of controlling objects with distributed parameters poses certain problems for researchers, which requires using non-standard research methods and decision-making in each specific case. Modern technical means of measuring and computing technology, which make it possible to carry out a large amount of measuring and computational work in real-time, have played a key role in the development of feedback control systems and their widespread practical implementation.

The paper considers the feedback control problem for the distributed parameters object on special classes of control actions. For synthesized controls, the concept of zonality is introduced, which means the constancy of the values of the synthesized control parameters in each of the subsets (zones), into which the entire set of possible states of the object is divided. The values of the control actions are determined by the type of feedback and the class of the functional dependence of the control on the observed value of the state. The constancy of the parameters of zonal control actions determines the robustness of the control system, as well as ensures the feasibility of synthesized control actions with sufficiently high accuracy and improves the technical performance of the equipment involved in the control loop.

We have used the principle of zonality of control parameters as the basis of numerical techniques for solving such specific optimization and inverse problems like the problem of optimal placement of production and injection wells and optimal control of their flow rates during the operation of an oil reservoir under the regime of water-driven piston displacement [1], the problem of identifying the hydraulic resistance coefficient under the unsteady flow of viscous fluids through pipelines [2], and problems of feedback control and identification of objects with lumped parameters [3–7].

II. Problem Statement

To illustrate the proposed approach, we consider the problem of controlling a rod heating process by means of lumped sources. This process can be described by the following parabolic type partial differential equation:

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}+\sum_{k=1}^{\ell}v_k(t)\delta(x-\xi_k)+\beta[u(x,t)-\gamma], \quad (1)$$

where $x\in(0,1)$, $t\in(0,T]$, $u(x,t)$ is the temperature of the rod at the point $x\in[0,1]$ at the moment of time $t\in[0,T]$; $\xi_k\in(0,1)$ the given locations of heat sources with optimizable powers $v_k(t)$, $k=1,2,...,\ell$; ℓ the number of heat sources; $\delta(.)$ the one-dimensional generalized Dirac's delta function; α the thermal diffusivity coefficient; β the heat transfer coefficient. Initial and boundary conditions are given in the following form:

$$u(x,0)=\varphi=\text{const}\ ,\ x\in(0,1), \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x}=\frac{\partial u(1,t)}{\partial x}=0\ ,\ t\in[0,T], \tag{3}$$

respectively. Note that the initial temperature φ, constant along the entire length of the rod, is not known exactly but there is given the set Φ of all possible values of the initial temperatures of the rod with a density function $\rho_\Phi(\varphi)$ such that

$$\rho_\Phi(\varphi)\ge 0\ ,\ \int_\Phi \rho_\Phi(\varphi)d\varphi=1. \tag{4}$$

The same is true concerning the parameter γ, the ambient temperature, whose exact values are not known but there is the set Γ of its possible values and the corresponding density function:

$$\rho_\Gamma(\gamma)\ge 0\ ,\ \int_\Gamma \rho_\Gamma(\gamma)d\gamma=1. \tag{5}$$

Assume that thermal sensors are installed at L points of the rod with coordinates x_i, $i=1,2,...,L$. These sensors are used to conduct operative observation and input to the control system of information on the state of the heating process at these points continuously in time:

$$\bar{u}(t)=\left(\bar{u}_1(t),...,\bar{u}_L(t)\right)=\left(u(x_1,t),...,u(x_L,t)\right), \tag{6}$$

or at discrete points in time:

$$\bar{u}(t_j)=\left(\bar{u}_1(t_j),...,\bar{u}_L(t_j)\right)=\left(u(x_1,t_j),...,u(x_L,t_j)\right), \tag{7}$$

for $j=0,1,2,...,N$. Based on technological conditions, we have to impose certain constraints on the values that the control actions can take:

$$U_k=\left\{v_k(t): v_k^{\min}\le v_k(t)\le v_k^{\max}, k=1,2,...,\ell\right\},\quad U=\left(U_1,U_2,...,U_\ell\right), \tag{8}$$

where $v_k^{\min}$ and $v_k^{\max}$ are prescribed values and U_k represents the set of admissible values of the control $v_k(t)$. Let the phase state values of the rod satisfy the inequalities:

$$u_{\min}\le u(x,t)\le u_{\max}, x\in(0,1), t\in[0,T],$$

under all possible admissible values of the controls $v_k(t)$, $k=1,2,...,\ell$, as well as initial conditions $\varphi\in\Phi$, boundary conditions (3), and ambient temperatures $\gamma\in\Gamma$. We divide the segment $[u_{\min},u_{\max}]$ by the points ω_s, $s=0,1,...,M$, such that $\omega_0=u_{\min}$, $\omega_M=u_{\max}$, and $\omega_{s-1}<\omega_s$, into semi-intervals $[\omega_{s-1},\omega_s)$, $s=1,2,...,M$. In the phase space of temperature values measured at the points x_i, $i=1,2,...,L$, of the rod at the current time t and some past time $t-\Delta$, we introduce L-dimensional parallelepipeds (so-called zones) as follows:

$$\begin{gathered}P^1_{i_1,i_2,...,i_L}=\left\{\left(\bar{u}_1(t),...,\bar{u}_L(t)\right):\omega_{i_s-1}\le u(x_s,t)\le\omega_{i_s}\right\},\\ P^2_{j_1,j_2,...,j_L}=\\ \left\{\left(\bar{u}_1(t-\Delta),...,\bar{u}_L(t-\Delta)\right):\omega_{j_s-1}\le u(x_s,t-\Delta)\le\omega_{j_s}\right\},\\ i_s, j_s\in\{1,2,...,M\}, s=1,2,...,L, P=(P^1_{i_1,...,i_L},P^2_{j_1,...,j_L}),\end{gathered} \tag{9}$$

where $I=(i_1,i_2,...,i_L;j_1,j_2,...,j_L)$ denotes the $2L$-dimensional multi-index, which indicates the number of the corresponding parallelepiped. The feedback control problem for the rod heating process consists in finding admissible values of the sources' powers as functions of the object's state:

$$v_k(t)=v_k\left(\bar{u}(t),\bar{u}(t-\Delta)\right), v_k(t)\in U_k, k=1,2,...,\ell,$$

at the observable points of the rod in order to minimize an objective functional. The source functions $v_k(t)$ are be assumed to be piecewise constant. The values of each control $v_k(t)$, constant for the whole time duration $[t_l,t_{l+1})$, are determined depending on the last measured value of the observation vector over the object's state $\left(\bar{u}(t),\bar{u}(t-\Delta)\right)$; namely, depending on the number (multi-index) of the parallelepiped (9), to which the last measured (observed) object's state $\left(\bar{u}(t),\bar{u}(t-\Delta)\right)$ belongs. Thus, to each phase parallelepiped (9), there corresponds its constant control value, i.e.,

$$\begin{gathered}v_k(t)=\vartheta_I^k=\text{const},\left(\bar{u}(t),\bar{u}(t-\Delta)\right)\in P_I,\\ t\in[t_l,t_{l+1}),\ l=0,1,2,...,N-1,\ k=1,2,...,\ell.\end{gathered} \tag{10}$$

In case the observed object's state belongs to the border of any zones, we use the value of the zonal control of that adjacent zone into which the phase trajectory has passed. The number of different values that each source's power can take is equal to the number of phase parallelepipeds, i.e., M^{2L}. It is clear that the controls of kind (10) assume feedback. In the case of (10), the values of the controllable sources' powers during the rod heating process change only at the moments when the population of states at the observable points proceeds from one phase parallelepiped to another. To control the heat conduction process in the rod, it is required to synthesize a regulator that, based on the results of temperature measurements at the points x_i, $i=1,2,...,L$, of the rod, would ensure the maintenance of the temperature $u(x,T)$ at a specified level by maintaining the required temperature $v_k(t)$, $k=1,2,...,\ell$, in the heat sources. In the case of non-fixed initial conditions and ambient temperature, the objective functional takes on the following form:

$$F(\vartheta)=\int_{\Phi}\int_{\Gamma}G(\vartheta;\varphi,\gamma)\rho_{\Gamma}(\gamma)\rho_{\Phi}(\varphi)\,d\gamma d\varphi,$$
$$G(\vartheta;\varphi,\gamma)=\int_0^1 w(x)\left[u(x,T;\vartheta,\varphi,\gamma)-u^*(x)\right]^2 dx, \quad (11)$$
$$\vartheta=\left(\vartheta_1^1,...,\vartheta_{M^{2L}}^1;\vartheta_1^2,...,\vartheta_{M^{2L}}^2;...;\vartheta_1^{\ell},...,\vartheta_{M^{2L}}^{\ell}\right),$$

where $u(x,T;\vartheta,\varphi,\gamma)$ is the solution to the initial- and boundary-value problem (1) – (3) corresponding to the initial condition $\varphi\in\Phi$, ambient temperature $\gamma\in\Gamma$, and to admissible values of the control $\vartheta\in U$; $w(x)$ the weight function; and $u^*(x)$ the function characterizing the desired distribution of temperature at the final moment of the heating process. The functional (11) characterizes the quality of the control process on average over the set of all possible initial states Φ, the set of possible ambient temperatures $\gamma\in\Gamma$, and the specified boundary conditions.

Thus, the considered control problem on the class of piecewise-constant functions with the use of feedback consists of optimizing the $\ell\times M^{2L}$-dimensional vector ϑ. The considered feedback control problem (1) – (11) is a parametric optimal control problem for a system with distributed parameters. Its specific features are, firstly, the absence of specifically prescribed initial conditions, secondly, the finite-dimensionality of the sought-for control vector, and thirdly, the control is formed depending on the values of the process state at the measurement points, and more precisely, it depends on the multi-index defining the parallelepiped (zone) of the phase space to which the measurement values belong. The solution of the feedback control problem (1) – (11) are synthesized zonal controls provided that the feedback with the object and the choice of the values of control actions is carried out only at specified discrete moments of time. As examples of practical applications of such problems, one can cite the control of many technological processes and technical objects. The organization of continuous monitoring of the state is impossible for these kinds of objects, and each observation (feedback) requires specific measures and, therefore, costs time and material.

The formulated feedback zonal controls problem (1) – (11) leads to a finite-dimensional optimization problem. For numerical solution to this problem, we propose to use the approach described in [2–7]. To solve the problem in the case of a simple design of a set of admissible controls U (for example, a parallelepiped, a hyper-sphere, a polyhedron, etc.), it is effective to use first-order numerical optimization methods such as gradient projection or conjugate gradient projection methods [8–11]. For example, for the conjugate gradient projection method, we construct a minimizing sequence $\{\vartheta^k\}$ in this fashion:

$$\vartheta^{k+1}=\mathrm{Proj}_{(U)}(\vartheta^k+\lambda_k\times d^k),\ \lambda_k>0,\ k=0,1,2,...,$$
$$d^0=-\nabla F(\vartheta^0),\ d^{k+1}=-\nabla F(\vartheta^{k+1})+\mu_k\times d^k, \quad (12)$$
$$\mu_k=\frac{\left\|\nabla F(\vartheta^{k+1})\right\|}{\left\|\nabla F(\vartheta^k)\right\|},$$

where the index k designates the iteration number; $\vartheta^0\in\mathbb{R}^{\ell\times M^{2L}}$ is some initial guess to the optimizable vector; $\nabla F(\vartheta^k)$ is the gradient of the objective functional; λ_k is the minimizing step size taken in the direction of d^k; $\mathrm{Proj}_{(U)}(\cdot)$ is the projection operator onto the admissible set U. If the domain U has a complex boundary and the projection operator onto it has no constructive character, then to solve the posed problem, one can use methods of sequential unconstrained optimization (for example, methods of internal and external penalty functions) with the use of effective methods of unconstrained optimization of the first order such as quasi-Newtonian methods [8, 9]. To construct iterative procedures based on the above optimization techniques, it is essential to have exact formulas for the gradient of the objective functional in the space of optimizable parameters. To this end, we derive formulas for the gradient of the objective functional in the space of optimizable parameters. The derivation of these formulas is based on the technique for calculating the increment of the objective functional obtained by incrementing the optimizable parameters.

In the derivation of the formula for the gradient of the objective functional, the following remark is important. The initial conditions (2), i.e., the elements of the set Φ, as well as different values of the parameter $\gamma\in\Gamma$ are independent. Then the gradient of the functional satisfies the formula:

$$\nabla F(\vartheta)=\nabla\int_{\Phi}\int_{\Gamma}G(\vartheta;\varphi,\gamma)\rho_{\Gamma}(\gamma)\rho_{\Phi}(\varphi)\,d\gamma d\varphi=$$
$$=\int_{\Phi}\int_{\Gamma}\nabla G(\vartheta;\varphi,\gamma)\rho_{\Gamma}(\gamma)\rho_{\Phi}(\varphi)\,d\gamma d\varphi.$$

Therefore, to obtain formulas for $\nabla F(\vartheta)$, we obtain formulas for the gradient of $G(\vartheta;\varphi,\gamma)$ with respect to individual terms φ and γ. For this purpose, we obtain the formula for the increment of the functional $G(\vartheta;\varphi,\gamma)$ obtained by incrementing a single component of the optimizable vector ϑ. Generalizing the formula for the gradient of $G(\vartheta;\varphi,\gamma)$ to all possible states of the initial condition and all possible values of the ambient temperature, i.e., covering the entire sets Φ and Γ, we thus prove the following theorem.

Theorem. The components of the gradient of the functional in the problem (1) – (11), in the space of piecewise constant controls (10) for an arbitrary control $\vartheta\in U$ are determined by the formula: as in

$$\frac{\partial F(\vartheta)}{\partial\vartheta_s^k}=-\int_{\Phi}\int_{\Gamma}\left\{\int_{\Pi_l(\vartheta_s^k)}\int_0^1\psi(x,t;\vartheta,\varphi,\gamma)\,\delta(x-\xi_k)\,dx\,dt\right\}\cdot$$
$$\cdot\rho_{\Gamma}(\gamma)\rho_{\Phi}(\varphi)\,d\gamma\,d\varphi,\ k=1,2,...,\ell,\ s=1,2,...,M^{2L},$$

where $\psi(x,t;\vartheta,\varphi,\gamma)$ is the solution of the following adjoint problem, corresponding to the current zonal control:

$$\beta\,\psi(x,t)-\frac{\partial\psi(x,t)}{\partial t}-\alpha\frac{\partial^2\psi(x,t)}{\partial x^2}=0, x\in(0,1), t\in[0,T)$$

$$\psi(x,T)=-2w(x)\left[u(x,T)-u^*(x)\right],\ x\in[0,1]$$

$$\frac{\partial\psi(0,t)}{\partial x}=\frac{\partial\psi(1,t)}{\partial x}=0,\ t\in[0,T].$$

Here $\Pi_I(\vartheta_s)=\bigcup_{\left(\bar{u}(t_j),\bar{u}(t_j-\Delta)\right)\in P_I}[t_j,t_{j+1})$ is the combined disparate time intervals, during which the object's state at the observable points belongs to the same zone, that is, when the vector $(\bar{u}(t_j),\bar{u}(t_j-\Delta))$ belongs to the I$^{\text{th}}$ phase parallelepiped. Note that the trajectory of the system may enter and leave the same zone multiple times.

The quality of the control system based on zonal control actions described above is significantly affected by choice of both the number of zones and their structure. Namely, an increase in the number of zones due to their refinement can only decrease the objective functional's value. So, an increase in the number of zones leads to a situation when control actions can change their values more often in time, and, therefore, on the one hand, the robustness of the control system deteriorates, and, on the other hand, this leads to rapid wear and failure of the actuating mechanisms. Conversely, an increase in the size of the zones, i.e., a decrease in their number, on the one hand, deteriorates the controllability of the object, and with a small number of them, the object may become completely uncontrollable. On the other hand, this increases the objective functional's value, i.e., the quality of control deteriorates. Taking these issues into account, the following approach is recommended, in which at first an initial value of M is arbitrarily selected and some zones are assigned. Having solved the feedback control problem, we can analyze the computed optimal zonal values of the controls for all neighboring zones. If the optimizable parameters in any two adjacent zones differ by a sufficiently small amount, then these adjacent zones can be combined into one, thus reducing the number M, the number of switchings of the control. If the optimizable parameters in any two adjacent zones differ significantly, then, on the contrary, each of these adjacent zones should be divided, for example, into two zones, i.e., increase the number M, and again solve the feedback control problem. An increase in the number of zones should be carried out until the objective functional's value ceases to change (decrease) significantly.

Remark 1. The frequency of observation times t_j, $j=0,1,2,...,N$, should be such that while the object's state belongs to any zone, at least one observation is made. If this condition is not met, the zones through which the system's trajectory did not pass under all possible initial conditions, as well as the zones through which no state measurements were carried out, will not be assigned the values of the zonal control parameters.

Remark 2. The main issue with the proposed approach to feedback is the high dimensionality of the optimizable control vector. The optimizable control vector's dimension represents a power function with respect to the number M of temperature intervals within the range of all possible temperature values of the object, and an exponential function with respect to the number of thermal sensors installed along the length of the rod. Besides, the number of thermal sources also affects the optimizable control vector (as a multiplication factor of the term M^{2L}). It is known that one of the basic problems of numerical optimization techniques (of any order) is the computation of optimal solutions of high-dimensional objective functions. This is because the optimization of high-dimensional objective functions is computationally expensive and cost involved, especially when seeking the global optimal solution. Many parameters characterize these kinds of problems, and many iterations and arithmetic operations are usually needed for evaluations of these objective functions. In order to speed up the evaluation of the objective functional in the posed feedback control problem, under the given value of the control vector, we can make use of the inherent concurrency present in the form of the objective functional. Namely, because the evaluation of the objective functional involves the computation of the definite integral, knowing that the elements of the sets Φ and Γ are independent, we can efficiently parallelize its computation by assigning to each thread (or process) a specific pair of elements (φ,γ), and computing the innermost definite integral in (11) with sufficiently high accuracy. The same concurrency pattern also applies to evaluating the gradient of the objective functional.

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