

On Stationary Probabilities of State-Dependent Markov Retrial Queue

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Oksana Pryshchepa

Department of Computer Science and Applied Mathematics
National University of Water and Environmental Engineering
Rivne, Ukraine
o.v.pryshchepa@nuwm.edu.ua

Abstract—In this paper, we consider the Markov model of a multiserver retrial queue with an input flow rate that depends on the number of calls in an orbit and with a limited number of retrials. We have obtained the conditions for the existence of the stationary regime and presented formulas for steady-state probabilities. Our approach is based on the approximation of the input model by the model with truncated state space.

Keywords—retrial queue; steady-state probabilities; service process.

I. INTRODUCTION

Retrial queues are a special class of stochastic models that take into account the features of the service process [1]. If the input call finds all servers busy, it becomes a source of repeated calls. The calls can try to get service an infinite number of times. However, it is possible that the number of repeated attempts can be limited. It might be if calls are impatient or resources are limited.

These models are widely used for the modelling of computer and telecommunication systems, call centres and control systems in airports. From a practical point of view, systems with variable input flow are very important and have to be tackled. The input flow rate in these systems depends on the number of repeated calls in the current moment of time [2]. This allows control of the input flow for maximizing the quality of service and maximizing the income from the system.

In this paper, we consider retrial queues with a controlled input flow rate and with a limited number of retrials. We have built explicit formulas of stationary probabilities using the truncated system.

II. MATHEMATICAL MODEL OF THE SYSTEM WITH A LIMITED NUMBER OF RETRIALS

Consider an $(m+1)$ -dimensional Markov chain $Q^m(t) = (Q_0(t), Q_1(t), \dots, Q_m(t))$, $t \geq 0$ in the phase space $S = \{0, 1, \dots, c\} \times Z_+^m$, where $Z_+ = \{0, 1, \dots\}$. $Q^m(t)$ is defined by its infinitesimal characteristics $q_{\beta\beta'}$, $\beta = (i, j_1, \dots, j_k, \dots, j_m)$, $\beta' = (i', j_1', \dots, j_k', \dots, j_m') \in S$:

- 1) if $i = 0, 1, \dots, c-1$, $j_k \in Z_+$, $k = 1, 2, \dots, m$, then

$$q_{\beta\beta'} = \begin{cases} \lambda_{j_1, \dots, j_k, \dots, j_m}, \beta' = (i+1, j_1, \dots, j_k, \dots, j_m), \\ i\mu, \beta' = (i-1, j_1, \dots, j_k, \dots, j_m), \\ j_1\nu_1, \beta' = (i+1, j_1-1, \dots, j_k, \dots, j_m), \\ j_2\nu_2, \beta' = (i+1, j_1, j_2-1, \dots, j_k, \dots, j_m), \\ \dots \\ j_k\nu_k, \beta' = (i+1, j_1, \dots, j_k-1, \dots, j_m), \\ \dots \\ j_m\nu_m, \beta' = (i+1, j_1, \dots, j_k, \dots, j_m-1), \\ - \left(\lambda_{j_1, \dots, j_k, \dots, j_m} + i\mu + \sum_{k=1}^m j_k\nu_k \right), \\ \beta' = (i, j_1, \dots, j_k, \dots, j_m), \\ 0, \text{ otherwise,} \end{cases}$$

- 2) if $i = c$, $j_k \in Z_+$, $k = 1, 2, \dots, m$, then

$$q_{\beta\beta'} = \begin{cases} \lambda_{j_1, \dots, j_k, \dots, j_m}, \beta' = (c, j_1+1, \dots, j_k, \dots, j_m), \\ c\mu, \beta' = (c-1, j_1, \dots, j_k, \dots, j_m), \\ j_1\nu_1, \beta' = (c, j_1-1, j_2+1, \dots, j_m), \\ j_2\nu_2, \beta' = (c, j_1, j_2-1, j_3+1, \dots, j_m), \\ \dots \\ j_k\nu_k, \beta' = (c, j_1, \dots, j_k-1, j_{k+1}+1, \dots, j_m), \\ \dots \\ j_{m-1}\nu_{m-1}, \\ \beta' = (c, j_1, \dots, j_k, \dots, j_{m-1}-1, j_m+1), \\ j_m\nu_m, \beta' = (c, j_1, \dots, j_k, \dots, j_m-1), \\ - \left(\lambda_{j_1, \dots, j_k, \dots, j_m} + c\mu + \sum_{k=1}^m j_k\nu_k \right), \\ \beta' = (c, j_1, \dots, j_k, \dots, j_m), \\ 0, \text{ otherwise.} \end{cases}$$

The component $Q_0(t)$ is the number of busy servers at the time $t \geq 0$ and $Q_k(t)$, $k=1, \dots, m$ is the number of retrial calls that have made k unsuccessful attempts to

get a service. In the retrial queue, the number of retrials is limited by m .

Markov chain $Q^m(t)$ models the service process in the following state-dependent retrial queue. System consists of c identical servers. From outside calls arrive for service. If there is at least one free server on the call arrival, it immediately gets service and then leaves the system. Service time is a exponentially distributed random variable with parameter μ . If all the servers are busy then the call creates a source of repeated calls and tries to get service in a random period. Each call is allowed to make m repeated attempts. If there are no free servers at the time of the last repeated attempt, the call abandons the system and does not get service.

The system is given by the following parameters: $\lambda_{j_1 \dots j_k \dots j_m}$ is the input flow rate, $j_k \in \mathbb{Z}_+$, $k=1, 2, \dots, m$; μ is the service rate; ν_k is k -th retrial rate, $k=1, 2, \dots, m$. Ergodicity conditions for the process $Q^m(t)$ are presented by the following lemma.

Lemma 1. If $\overline{\lim}_{j_m \rightarrow \infty} j_m^{-1} \lambda_{j_1 \dots j_k \dots j_m} < \nu_m$ and $\lambda_{j_1 \dots j_k \dots j_m}, \mu, \nu_k > 0, j_k \in \mathbb{Z}_+, k=1, 2, \dots, m$, then $Q^m(t)$ is ergodic and its ergodic distribution $\pi_{i_{j_1 \dots j_k \dots j_m}}, (i, j_1, \dots, j_m) \in S$.

In order to obtain a representation of $\pi_{i_{j_1 \dots j_k \dots j_m}}, (i, j_1, \dots, j_m) \in S$ in terms of system parameters, we use approximation with truncated system technique. This system has a fixed number of N places in each set of retrials. This means that the call leaves the system after k -th attempt to get service if all servers are busy and there are N calls that have made k attempts already. The service process for $m \geq 2$ becomes complicated and explicit formulas of the stationary probabilities have not been found so far. However, for $m=1$ we give a representation of $\pi_{i_{j_1 \dots j_k \dots j_m}}, (i, j_1, \dots, j_m) \in S$ in terms of system parameters in an explicit form.

III. STATIONARY PROBABILITIES

Theorem 1. Let for the one-channel state-dependent retrial queue with one retrial the conditions of Lemma 1 hold. Then, for the service process $Q^1(t) = (Q_0(t), Q_1(t))$, a stationary regime exists and stationary probabilities take the form:

$$\pi_{0j} = \frac{1}{j! \nu_1^j} \prod_{i=1}^j \frac{\lambda_{i-1} (\lambda_{i-1} + (i-1) \nu_1)}{\lambda_i + i \nu_1 + \mu} \pi_{00}, \varphi=1, 2, \square,$$

$$\pi_{1j} = \frac{\lambda_0}{\mu} \frac{1}{j! \nu_1^j} \prod_{i=1}^j \frac{\lambda_{i-1} (\lambda_i + i \nu_1)}{\lambda_i + i \nu_1 + \mu} \pi_{00}, \varphi=0, 1, \square,$$

where

$$\pi_{00}^{-1} = \sum_{j=0}^{\infty} \frac{1}{j! \nu_1^j} \prod_{i=1}^j \frac{\lambda_{i-1} (\lambda_{i-1} + (i-1) \nu_1)}{\lambda_i + i \nu_1 + \mu},$$

$$+ \frac{\lambda_0}{\mu} \sum_{j=0}^{\infty} \frac{1}{j! \nu_1^j} \prod_{i=1}^j \frac{\lambda_{i-1} (\lambda_i + i \nu_1)}{\lambda_i + i \nu_1 + \mu}.$$

Let us introduce $x_j = \sum_{n=1}^{\infty} \frac{\alpha_{j+n}}{\beta_{j+n}}$ be the continued fraction where for $j = 1, 2, \dots$

$$\alpha_j = - \frac{\lambda_{j-1} ((\lambda_{j-1} + (j-1) \nu_1)^2 + (j-1) \nu_1 \mu)}{j(j+1) \nu_1^2 \mu},$$

$$\beta_j = - \frac{j \nu_1 ((\lambda_j + \mu + j \nu_1)^2 + \mu (\lambda_{j-1} + \mu + j \nu_1))}{j(j+1) \nu_1^2 \mu}.$$

Theorem 2. Let for the two-channel state-dependent retrial queue with one retrial the conditions of Lemma 1 hold. Then, for the service process, a stationary regime exists and stationary probabilities take the form:

$$\pi_{0j} = \pi_{00} \prod_{k=0}^{j-1} x_k, \quad j = 1, 2, \dots,$$

$$\pi_{1j} = \frac{(\lambda_j + j \nu_1) \pi_{00}}{\mu} \prod_{k=0}^{j-1} x_k, \quad j = 0, 1, \dots,$$

$$\pi_{2j} = \frac{1}{2\mu^2} ((\lambda_j + j \nu_1)^2 + j \nu_1 \mu - (j+1) \nu_1 \mu x_j) \times$$

$$\times \pi_{00} \prod_{k=0}^{j-1} x_k, \quad \varphi=0, 1, \square,$$

where

$$\pi_{00}^{-1} = \frac{1}{2\mu^2} \times$$

$$\times \left(\sum_{j=0}^{\infty} ((\lambda_j + \mu + j \nu)^2 + \mu(\mu + j \nu - (j+1) \nu x_j)) \prod_{k=0}^{j-1} x_k \right).$$

The obtained results can be used to solve optimization problems in the class of threshold strategies [3].

CONCLUSION

In this paper, we have presented state-dependent retrial queues with a limited number of repeated attempts and variable input flow that depends on the current number of repeated calls. For this type of system, we have found ergodicity conditions and obtained explicit representations of steady-state probabilities. These results allow us to analyse the service process, compute the characteristics of the systems and solve optimization problems.

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