Some approximate properties of Cesaro means
Fourier series

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Abstract— It is established \( \sigma_\alpha^*(x, f) \) Cesaro means of some approximate features. The paper discusses the new property of the negative order Cesaro mean of the Fourier trigonometric series.

Keywords— Cesaro means; Fourier trigonometric series.

I. INTRODUCTION

Suppose, that \( T = [-\pi, \pi] \) and \( f : \mathbb{R} \rightarrow \mathbb{R} \) 2\( \pi \) - are periodic functions. If \( f \in L(T) \) as a rule, \( \sigma(f) \) represent respectively trigonometric Furrier series

\[
\sigma[f](x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i \cos kx + b_i \sin kx,
\]

where

\[
a_i = a_i(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad k \in \mathbb{N},
b_i = b_i(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt, \quad k \in \mathbb{N}.
\]

Assume that \( p \in [1, +\infty] \) - is a number. For each function \( f \in L^p(T) \) the following will be considered

\[
\|f\|_p = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{\frac{1}{p}},
\]

And also it will consider the following:

\[
L^\infty(T) = c(T), \quad \|f\|_\infty = \sup_{x \in T} |f(x)|. \quad \text{Let us set}
\]

\[
\omega^{(k)}(\sigma, f) = \sup_{|\theta| < \frac{\pi}{2}} \left| \sum_{j=0}^{\infty} (-1)^{j+1} \left( \int f(t+jh) \right) \right|_{\theta, p}.
\]

\( \omega^{(k)}(\sigma, f) \) is called module \( L' \) of smoothness of an order to function \( f \).

II. MAIN PART

In the future we will assume that \( \omega^{(k)}(\sigma, f), p = \omega(\sigma, f), p \).

BelowA, \( A(f), A(f, p), A(f, a, p), A(a), A_h(a), ... \) indicate absolute positive or positive constants depending only on the specified parameters.

Let \( \omega \)-be module of continuity. Assume that \( H^\infty = H_p(T) = \{ f : \omega(\sigma, f), p \leq A(f) \omega(\sigma) \} \)

and \( H^\infty = H_p \). If \( \omega(\sigma) = \sigma^\alpha, \quad \alpha \in [0,1] \), then \( H^\infty = \text{Lip}(a, p) \). \( H^\infty = \text{Lipa}. \)

With \( S_\alpha(x, f) \) we will define, respectively partial sums of the series

\[
S_\alpha(x, f) = \frac{a_0}{2} + \sum_{i=1}^{n} a_i \cos kx + b_i \sin ks =
\]

\[
= \frac{1}{\pi} \int f(x+t) D_k(t) dt,
\]

where \( D_k(t) \) is Dirichlet kernel, i.e.

\[
D_k(t) = \frac{1}{2^n} \sum_{i=1}^{2^n} \cos k \sin (n + \frac{1}{2}) t / 2 sin \frac{t}{2^n}.
\]

and \( D_k(0) = n + 0.5, \quad n \in \mathbb{N} \).

Consider that

\[
\mathcal{A}_n = 1, \quad \mathcal{A}_n = \frac{(a+1)(a+2)(a+3)......(a+k)}{k!},
\]

\( k \in \mathbb{N}, \quad a > -1 \).

It is known (see e.g. A. Sigmund [7], pp. 130-131) that
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\[ A^n_i = \sum_{i=0}^{k} A^n_{i+1}, \ A^n_i - A^n_{i-1} = A^n_i \quad (7) \]

\[ A(a) \leq A^n_i / k_n \leq A_1(a). \quad (8) \]

If

\[ K^n_i(t) = \frac{1}{A^n_i} \sum_{i=0}^{k} A^n_{i+1} D_i(t). \quad (9) \]

that they are respectively called Chesaro kernel. It is known that (see e.g. A. Sigmund [2] pp. 157-164), that

\[ K^n_i(t) = \phi^n_i(t) + \sigma_{\alpha}(t). \quad (10) \]

where

\[ \phi^n_i(t) = \sin \left( \frac{n + \frac{1}{2} + a}{2} t - \frac{a \pi}{2} \right) A^n_i \left( 2 \sin \frac{t}{2} \right)^{i+a} \quad (11) \]

and

\[ \left\| K^n_i \right\| \leq A(a)n, \quad (12) \]

\[ \left\| \phi^n_i(t) \right\| \leq \frac{A(a)}{nt} - \frac{\pi}{n} \quad \text{if} \quad t \leq \pi. \quad (13) \]

In what follows we shall use the Holder inequality for integrals. If \( f_i \in L^p(T), \ f_2 \in L^q(T) \) and \( 1/p + 1/q = 1, \)

\[ \left\| f_i f_2 \right\| \leq \left\| f_i \right\| \left\| f_2 \right\| _q. \quad (14) \]

We also use the Minkowski inequality. If \( f_i \in L^p(T), \ f_2 \in L^q(T), \ p \in [1, +\infty] \), then

\[ \left\| f_i + f_2 \right\| \leq \left\| f_i \right\| + \left\| f_2 \right\| _q. \quad (15) \]

Assume that function \( f \in L(T) \) then

\[ \sigma^n_{\alpha}(x, f) = \frac{1}{A^n_i} \sum_{i=0}^{k} A^n_{i+1} S_i(x, f), \quad (16) \]

where \( A^n_i \) \( (i = 1, 2, \ldots) \) and \( S_i(x, f) \) are given to relations (6) and (4). Using the equalities (4), (5), (9) and (16), we can write

\[ \sigma^n_{\alpha}(x, f) = \frac{1}{\pi} \int f(x+t)K^n_i(t)dt = \quad (17) \]

\[ = \frac{1}{\pi} \int \left[ f(x+t) + f(x-t) \right]K^n_i(t)dt \]

Zigmund proved that if the function \( f \in c(T) \) and

\[ \omega(\delta, f)_{\alpha} = \overline{D}(\delta^{\alpha}) (\delta \to +0), \ \alpha \in [0; 1], \quad (18) \]

then

\[ \lim_{n \to \infty} \left\| \sigma^n_{\alpha}(f) - f \right\| = 0. \quad (19) \]

In the writings of Hardy and Littlewood mentioned provision was generalized, in particular, they showed that if \( f \in c(T) \) and

\[ \omega(\delta, f)_{p} = \overline{D}(\delta^{\alpha}) (\delta \to +0), \ \alpha \in [0; 1], \ \alpha p > 1, \quad (20) \]

then it is executed (19). They showed that when \( f \in c(T) \cap \text{Lip}(a, p), \ \alpha \in [0; 1], \ \alpha p > 1, \)

Then for any \( \beta \to 0; \alpha \) numbers the following ratio is valid yeah

\[ \lim_{n \to \infty} \left\| \sigma^n_{\alpha, \beta}(f) - f \right\| = 0. \quad (21) \]

Our goal is to evaluate the above expression

\[ \left\| \sigma^n_{\alpha}(x, f) - f \right\| . \]

With the help of the modules of continuity of space \( c(T) \) and \( L^p(T) \). These estimates will be taken some new position in relation to the behavior \( \sigma^n_{\alpha}(x, f) \) means.

III. CONCLUSION

The following in true

Theorem.

Let \( a \in [0, 1] - \) a certain number \( ap = 1, \ \beta \in [0, 1], \) and \( 0 < \lambda < 0.5. \)

If \( \lambda \geq 1/(1+a) \) then for the function \( f \in c(T) \) the following inequality is true

\[ \left\| \sigma^n_{\alpha, \beta}(f) - f \right\| \leq A(a, \beta) \left[ \omega^{(a, \beta)}(\frac{1}{n}, f) + \frac{1}{1+n^\alpha} \omega^{(a, \beta)}(\frac{1}{n}, f) \right] + A(\beta) g(n, f) \]

The given theorem represents a new property of the Cesaro mean of the negative order of the trigonometric Fourier series.

REFERENCES