Some approximate properties of Cesaro means Fourier series

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Abstract— **It is established $\sigma_n^a(x,f)$ Cesaro means some approximate features. The paper discusses the new property of the negative order Cesaro mean of the Fourier trigonometric series.**

Keywords— Chesaro means; Fourier trigonometric series.

I. Introduction

Suppose, that $T=[-\pi,\pi]$ and $f:R\to R$ 2π - are periodic functions. If $f\in L(T)$ as a rule, $\sigma(f)$ represent respectively trigonometric Furrier series

$$\sigma[f](x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx+b_k\sin kx, \qquad (1)$$

where

$$\left.\begin{aligned} a_k\equiv a_k(f)=\frac{1}{\pi}\int_T f(t)\cos kt\ \ dt,\ \ k\in N_0\\ b_k=b_k(f)=\frac{1}{\pi}\int_T f(t)\sin kt\ \ dt,\ \ k\in N.\end{aligned}\right\} \qquad (2)$$

Assume that $p\in[1,+\infty[$ - is a number. For each function $f\in L^p(T)$ the following will be considered

$$\|f\|_p=\left\{\frac{1}{2\pi}\int_T|f(t)|^p\,dt\right\}^{\frac{1}{p}}, \qquad (3)$$

And also it will consider the following: $L^\infty(T)=c(T)$, $\|f\|_c=\|f\|_\infty=\sup_{x\in T}|f(x)|$. Let us set

$$\omega^{(k)}(\sigma,\ f)=\sup_{|h|\le\sigma}\left\|\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}f(t+jh)\right\|_p;$$

$$\sigma\in\,]0,2\pi].$$

$\omega^{(k)}(\sigma,f)$ is called module L^p of smoothness of an order to function f.

II. Main part

In the future we will assume that $\omega^{(1)}(\sigma,f)_p\equiv\omega(\sigma,f)_p$.

BellowA, $A(f)$ $A(f,p)$, $A(f,a,p)$, $A(a)$, $A_1(a)$, ... indicate absolute positive or positive constants depending only on the specified parameters.

Let ω -be module of continuity. Assume that

$$H_p^\omega\equiv H_p^\omega(T)=\{f:\omega(\sigma,f)_p\le A(f,p)\omega(\sigma)\}$$

and $H_p^\omega=H^\omega$. If $\omega(\sigma)=\sigma^\alpha$, $\alpha\in\,]0,1]$, than $H_p^\omega\equiv Lip(a,p)$. $H^\omega\equiv Lipa$.

With $S_n(x,f)$ we will define, respectively partial sums of the series

$$\begin{aligned}S_n(x,f)&=\frac{a_0}{2}+\sum_{k=1}^{n}a_k\cos kx+b_k\sin ks=\\ &=\frac{1}{\pi}\int_T f(x+t)D_n(t)dt,\end{aligned} \qquad (4)$$

where $D_n(t)$ - is Dirichlet kernel, i.e.

$$D_n(t)=\frac{1}{2}+\sum_{k=1}^{n}\cos kt=\sin\left(n+\frac{1}{2}\right)t/2\sin\frac{t}{2}, \qquad (5)$$

and $D_n(0)=n+0.5,\ \ n\in N$.

Consider that

$$A_k^a=1,\ \ A_k^a=\frac{(a+1)(a+2)(a+3)......(a+k)}{k!}, \qquad (6)$$

$$k\in N,\ \ a>-1.$$

It is known (see e.g. A. Sigmund [7], pp. 130-131) that

$$A_k^a = \sum_{i=0}^{k} A_{k-i}^{a-1}, \; A_k^a - A_{k-1}^a = A_k^{a-1} \tag{7}$$

$$A(a) \le \frac{A_k^a}{k^a} \le A_1(a). \tag{8}$$

If

$$K_n^a(t) = \frac{1}{A_n^a} \sum_{k=0}^{n} A_{n-k}^{a-1} D_k(t), \tag{9}$$

than they are respectively called Chesaro kernel. It is known that (see e.g. A. Sigmund [2] pp. 157-164), that

$$K_n^a(t) = \varphi_n^a(t) + r_n^a(t), \tag{10}$$

where

$$\varphi_n^a(t) = \frac{\sin\left[\left(n + \frac{1}{2} + \frac{a}{2}\right)t - \frac{a\pi}{2}\right]}{A_n^a \left(2\sin\frac{t}{2}\right)^{1+a}} \tag{11}$$

and

$$\left\|K_n^a\right\|_c \le A(a)n, \tag{12}$$

$$\left|\tau_n^a(t)\right| \le \frac{A(a)}{nt^2}, \; \frac{\pi}{n} \le t \le \pi, \tag{13}$$

In what follows we shall use the Holder inequality for integrals. If $f_1 \in L^p(T)$, $f_2 \in L^q(T)$ and $\frac{1}{p} + \frac{1}{q} = 1$, than

$$\| f_1 f_2 \|_L \le \| f_1 \|_p \| f_2 \|_q . \tag{14}$$

We also use the Minkowski inequality. If $f_1 \in L^p(T)$, $f_2 \in L^p(T)$, $p \in [1, +\infty[$, than

$$\| f_1 + f_2 \|_p \le \| f_1 \|_p + \| f_1 \|_p \tag{15}$$

Assume that function $f \in L(T)$ than $\sigma_n^a(x, f)$ $t_n^a(x, f)$ symbols denotes the Chesaro means of order $a > -1$ consequently $\sigma[f]$ i.e.

$$\sigma_n^a(x, f) = \frac{1}{A_n^a} \sum_{k=0}^{n} A_{n-1}^{a-1} S_k(x, f), \tag{16}$$

where A_n^α $(i = 1, 2, ...)$ and $S_k(x, f)$ are given to relations (6) and (4). Using the equalities (4), (5), (9) and (16), we can write

$$\sigma_n^\alpha(x, f) = \frac{1}{\pi} \int_T f(x+t) K_n^\alpha(t) dt = \\ = \frac{1}{\pi} \int_T [f(x+t) + f(x-t)] K_n^\alpha(t) dt \tag{17}$$

Zigmund proved that if the function $f \in c(T)$ and

$$\omega(\delta, f)_c = \bar{\bar{\text{O}}}(\delta^\alpha)(\delta \to +0), \;\; \alpha \in]0;1[, \tag{18}$$

then

$$\lim_{n\to\infty} \| \sigma_n^{-\alpha}(f) - f \|_c = 0. \tag{19}$$

In the writings of Hardy and Littlewood mentioned provision was generalized, in particular, they showed that if $f \in c(T)$ and

$$\omega(\delta, f)_p = \bar{\bar{\text{O}}}(\delta^\alpha)(\delta \to +0), \;\; \alpha \in]0;1[, \;\; \alpha p > 1, \tag{20}$$

then it is executed (19). They showed that when

$$f \in c(T) \cap \text{L}ip(\alpha, p), \;\; \alpha \in]0;1[, \;\; \alpha p > 1,$$

Then for any $\beta \in]0; \alpha[$ numbers the following ratio is valid yeah

$$\lim_{n\to\infty} \| \sigma_n^{-\beta}(f) - f \|_c = 0. \tag{21}$$

Our goal is to evaluate the above expression

$$\| \sigma_n^\alpha(x, f) - f \|_c .$$

With the help of the modules of continuity of space $c(T)$ and $\text{L}^p(T)$. These estimates will be taken some new position in relation to the behavior $\sigma_n^\alpha(x, f)$ means.

III. Conclusion

The following in true

Theorem.

Let $a \in]0,1]$ - a curtain number $ap = 1$, $\beta \in]0, a[$ and $0 < \lambda < 0.5$.

If $\lambda \ge 1/(1+a)$ than for the function $f \in c(T)$ the following inequality is true

$$\left\|\sigma_n^{-\beta}(f) - f\right\|_c \le A(a, \beta)\left[\omega^{\lambda(a-\beta)}\left(\frac{1}{n}, f\right)_c\right] + \\ + \left[1 + n^a \omega\left(\frac{1}{n}, f\right)_p\right] + A(\beta) g(n, f)$$

The given theorem represents a new property of the Cesaro mean of the negative order of the trigonometric Fourier series.

References

[1] A. Zigmund, “Trigonometric series,” 3rd ed., Cambridge: Cambridge University Press, 1965.

[2] G.G. Xardi, D.E. Littlvud, and G. Polya, “Inequalities,” 2nd ed., Cambridge: Cambridge University Press, 2001.

[3] L.V. Zhizhiashvili, “Some problems about trigonometric series and its conjugates,” Tbilisi: Tbilisi University Press, 1993 (in Russian).

[4] M.S. Bakuridze, “Some approximate properties of Cesaro mean Fourier series and their conjugates,” International Journal of Engineering Science and Innovative Technology, vol. 4(6), 2015.

[5] M.S. Bakuridze, “Some approximate properties of Cesaro mean Fourier series and their conjugates,” International Journal of Engineering Science and Innovative Technology vol. 5(4), 2016.

[6] M.S. Bakuridze and A.S. Bakuridze, “Some approximate properties of Cesaro mean Fourier series,” International Journal of Engineering Science and Innovative Technology, vol. 7(3), 2018.