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Point Optimal Control of Pseudoparabolic Systems with Memory

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Abstract **— We use the method of a priori inequalities in negative norms to prove the well-posedness of the Dirichlet initial boundary value problem for the differential and integro-differential equations of the pseudo-parabolic type with integral terms of the Volterra type. Further we provide theorem on optimal control existence. The result is obtained under less assumptions on the equation's coefficients than in similar papers.**

Keywords: **Dirichlet problem, integro-differential equation, Volterra operator, a priori estimates, generalized solutions**

I. INTRODUCTION

Pseudoparabolic differential and integro-differential models are among the many objects studied in applied mathematics. Models of this type describe a wide class of physical processes, including: radiation with time delay, two-phase models of porous media flow with dynamic capillarity or hysteresis, ion migration in soil, heat conduction in heterogeneous media, etc. (see, for example, [1],[2] and the references therein).

In [1],[3] S.I. Lyashko and his colleagues developed a method of a priori inequalities in negative norms, by which a wide range of optimization problems for various systems with distributed parameters, including differential models of the pseudoparabolic type, were studied (look also [4] and the references there).

Subsequently, it turned out that this approach could also be effectively applied to Dirichlet problems for integro-differential equations with Volterra-type integral components [2], [5], [6].

In our work, we prove a priori inequalities similar to those in [1] and [2] for the linear pseudoparabolic differential equation

$$\mathcal{L}_1 u \equiv -\sum_{i,j=1}^{n}\left(a_{ij}(x)\,u_{x_j}\right)_{x_i t} + a(x)u_t - -\sum_{i,j=1}^{n}\left(b_{ij}(x)\,u_{x_j}\right)_{x_i} + b(x)u = f(x,t), \quad (1)$$

and the linear integro-differential equation with a Volterra-type integral operator

$$\mathcal{L}_2 u \equiv \mathcal{L}_1 u + + \int_0^t \sum_{i=1}^{n}\left(K_i(x,t,\tau)\,u_{x_i}(x,\tau)\right)_{x_i} d\tau = f(x,t), \quad (2)$$

with initial-boundary Dirichlet conditions

$$u|_{t=0} = 0,\ u|_{x\in\partial\Omega} = 0. \quad (3)$$

At the same time, we do not require the conditions of non-negative definiteness of the coefficient matrix $\{b_{ij}\}_{i,j=1}^{n}$ and the non-negativity of the function b, which are required in [1] and [2]. Thus, the results concerning the well-posedness of the initial-boundary problems and the existence of optimal control will be justified under weaker requirements for the equation coefficients.

II. MAIN NOTATION

Let the system's evolution be described by the linear equation $\mathcal{L}u = f$, where $\mathcal{L}$ can denote either the differential operator (1) or the integro-differential operator (2). The function $u(x,t)$ describes the state of the system in the domain $Q = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded spatial domain with a smooth boundary $\partial\Omega$. The function u satisfies the homogeneous initial-boundary Dirichlet conditions (3).

We assume that $\{a_{ij}\}_{i,j=1}^{n}, \{b_{ij}\}_{i,j=1}^{n} \subset C^1(\overline{\Omega}), a, b \in C(\overline{\Omega})$, the kernels $K_i(x,t,\tau)$ are continuously differentiable, and for all $x \in \Omega$ the following relations hold

$$a_{ij}(x) = a_{ji}(x),\ b_{ij}(x) = b_{ji}(x),\ a(x) \ge 0,$$

and there exists a positive constant α such that the functions $a_{ij}(x)$ satisfy the condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \alpha \sum_{i=1}^{n} \xi_i^2,$$

for any $x \in \Omega$ and $\xi_i \in \mathbb{R}$, $i = \overline{1,n}$. Notably, unlike in [1],[2], we do not require other conditions on the coefficients b_{ij}, b.

We consider the set C_{BR}^{∞} of infinitely differentiable functions in the domain $\overline{Q}$ that satisfy the homogeneous initial and boundary conditions (3), and the set $C_{BR^+}^{\infty}$ of infinitely differentiable functions $v(x,t)$ that satisfy the conditions

$$v|_{t=T} = 0,\ v|_{x\in\partial\Omega} = 0. \quad (4)$$

The domains of definition of the operators $\mathcal{L}$ and $\mathcal{L}^*$ will be considered as the spaces C_{BR}^{∞} and $C_{BR^+}^{\infty}$, respectively.

By W_{BR}, H_{BR} we denote the completion of the space of smooth functions C_{BR}^{∞} that satisfy the conditions (3) with respect to the norms

$$\| u \|_{W_{BR}} = \left(\int_Q u_t^2 + \sum_{i=1}^{n} u_{x_i t}^2 \, dQ \right)^{\frac{1}{2}}, \qquad (5)$$

$$\| u \|_{H_{BR}} = \left(\int_Q u^2 + \sum_{i=1}^{n} u_{x_i}^2 \, dQ \right)^{\frac{1}{2}}. \qquad (6)$$

Similarly, let W_{BR^+}, H_{BR^+} be the completion of the space of smooth functions $C_{BR^+}^{\infty}$ with respect to the norms (5), (6). Finally, by $W_{BR}^-, H_{BR}^-, W_{BR^+}^-, H_{BR^+}^-$ we denote the negative spaces constructed from the corresponding positive spaces with respect to $L_2(Q)$.

III. MAIN RESULTS

The following theorem holds:

Theorem 1. *There exist constants $C_1 > 0$, $C_2 > 0$ such that for any $u \in W_{BR}$ and $v \in W_{BR^+}$ the inequalities hold:*

$$\begin{cases} C_1 \| u \|_{H_{BR}} \leq \| \mathcal{L}u \|_{W_{BR^+}^-} \leq C_2 \| u \|_{W_{BR}}, \\ C_1 \| v \|_{H_{BR^+}} \leq \| \mathcal{L}^* v \|_{W_{BR}^-} \leq C_2 \| v \|_{W_{BR^+}}. \end{cases} \qquad (7)$$

We use the following definition of a solution of the equation $\mathcal{L}u = f$

Definition. *A function $u \in H_{BR}$ is called a generalized (weak) solution of the equation $\mathcal{L}u = f$, $f \in W_{BR^+}^-$ if*

$$\langle u, \mathcal{L}^* v \rangle_{H_{BR}} = \langle f, v \rangle_{W_{BR^+}},$$

for any functions $v \in W_{BR^+}$ such that $\mathcal{L}^* v \in H_{BR}^-$.

Based on the inequalities (7) and the results of the works [1],[2], we formulate the theorem on the problem well-posedness:

Theorem 2. *For any $f \in W_{BR^+}^-$ there exists a unique solution $u \in H_{BR}$ of the equation $\mathcal{L}u = f$ in the sense of definition above, and there exists a constant $C > 0$ such that $\| u \|_{H_{BR}} \leq C \| f \|_{W_{BR^+}^-}$.*

Remark. *Similar definition and theorem could be formulated for the adjoint problem.*

Let us consider the problem of optimal control of a system, the evolution of which is described by the linear equation

$$\mathcal{L}u = f + \mathcal{A}h, \; f \in W_{BR^+}^-. \qquad (8)$$

Here $u(x,t) \in H_{BR}$ is a function that dependins on the control h, defined in the admissible set $\mathcal{U}$ of the control space $\mathcal{H} = \left([a,b] \times L_2\left((0,T) \times \Omega'\right)\right)^s$, and

$$\mathcal{A}h = \sum_{i=1}^{s} \delta\left(x_1 - x_{1,i}\right) \otimes \varphi_i(t, x_2, \ldots, x_n). \qquad (9)$$

Here

$$x_1, x_{1,i} \in [a,b], \; \varphi_i(t, x_2, \ldots, x_n) \in L_2\left((0,T) \times \Omega'\right).$$

On the solutions of the equation (8) a certain functional $J(h) = \Phi\left(u(h)\right)$ is defined, which needs to be minimized under the condition $h \in \mathcal{U}$.

The following theorem on the existence of optimal control holds

Theorem 3. *If the quality criterion $\Phi: H_{BR} \to \mathbb{R}$ is weakly lower semi-continuous with respect to the system state $u(x,t,h)$ and is bounded from below, and if the set of admissible controls $\mathcal{U}$ is closed, bounded, and convex in $\mathcal{H}$, then the optimal control of the system (8) with the control operator (9) exists.*

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