Dynamical Picture of Biological Populations with Attractive and Repulsive Interaction

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Abstract— **We study a model of discrete conflict dynamical system with attractive and repulsive interaction, which describes the redistribution of common space of existence between three alternative populations. The evolution of the redistribution of common space of existence is illustrated by examples. It is assume that two populations struggle to survive on a shared space of existence (repulsive interaction), and third coexist with first two ones in the same space (attractive interaction).**

Keywords— dynamical conflict systems; fixed point; limit state, attractive and repulsive interaction.

I. Introduction

We consider a model of complex dynamical system describing the conflict redistribution of vital resource between three populations A, B, C (compare with [2, 3, 7]).

In our approach each state of the system at discrete time moments $t = 0, 1, \ldots$ is described by their densities $\rho_i^t(x)$, $i = 1, 2, 3$ on a compact Ω (or by stochastic vectors $\mathbf{p}_i^t$, $i = 1, 2, 3$) according to equations

$$\rho_i^{t+1}(x) = \rho_i^t(x)(\theta^t + 1) \pm \kappa_i^t(x), \tag{1}$$

or for vectors in terms of coordinates

$$p_i^{t+1} = p_i^t(\theta^t + 1) \pm \tau_i^t. \tag{2}$$

We assume that all biological populations are non-annihilating and moreover have probabilistic interpretation. We are interesting in time asymptotical behavior [5]

$$\lim_{t\to\infty} \rho_i^t(x) = \rho_i^\infty(x) \quad (\lim_{t\to\infty} p_i^t = p_i^\infty).$$

We start with simplest vector version of the Lotka-Volterra equation with repulsive and attractive interaction between three biological populations. To each population we associate one of stochastic vectors $\mathbf{p}, \mathbf{r}, \mathbf{q} \in \mathbb{R}_+^n$, $n > 1$, $\|\mathbf{p}\|_1 = \|\mathbf{r}\|_1 = \|\mathbf{q}\|_1 = 1$, where $\|\cdot\|_1$ denote l_1 norm on $\mathbb{R}_+^n$. The evolution of $\mathbf{p}, \mathbf{r}, \mathbf{q}$ in time under the conflict interaction is governed by the simplest vector version of the Lotka-Volterra equations

$$\begin{cases} \dot{\mathbf{p}} = \mathbf{p} * (1 - \mathbf{p}), \\ \dot{\mathbf{r}} = \mathbf{r} * \left(1 + \dfrac{1}{2}(\mathbf{p} + \mathbf{q})\right), \\ \dot{\mathbf{q}} = \mathbf{q} * (1 - \mathbf{q}). \end{cases} \tag{3}$$

II. The Conflict Dynamical System

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, $n > 1$ denote a finite space of controversial positions for three populations, which are represented by discrete probability measures μ_1, μ_2 and μ_3 on Ω. The starting distributions of μ_1, μ_2 and μ_3 along Ω are given by three sets of numbers

$$p_i := \mu_1(\omega_i), \qquad \sum_{i=1}^n p_i = 1,$$

$$r_i := \mu_2(\omega_i), \qquad \sum_{i=1}^n r_i = 1,$$

$$q_i := \mu_3(\omega_i), \qquad \sum_{i=1}^n q_i = 1.$$

So, the vectors $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $\mathbf{r} = (r_1, r_2, \ldots, r_n)$, $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ from $\mathbb{R}_+^n$ are stochastic, $0 \le p_i, r_i, q_i \le 1$, $i = \overline{1, n}$, and further we assume that they are non-orthogonal.

Values p_i, r_i, q_i have the probabilistic sense for random visits of positions ω_i by population A, B and C, respectively. So one can write, $p_i := \mathbf{P}(B$ is in $\omega_i)$, where $\mathbf{P}(\cdot)$ means a probability, and similar for r_i and q_i. Assume, that at next moments of discrete time $t = 1, 2, \ldots$ the population A, B and C begin to interact one to other according the law

$$\begin{aligned} p_i^{t+1} &= \frac{1}{z_-^t} p_i^t(1 - q_i^t), \\ r_i^{t+1} &= \frac{1}{z_+^t} r_i^t(1 + \tau_i^t), \quad \tau_i^t = \frac{p_i^t + q_i^t}{2}, \\ q_i^{t+1} &= \frac{1}{z_-^t} q_i^t(1 - p_i^t), \quad t = 0, 1, 2, \ldots, \end{aligned} \tag{4}$$

where $p_i^0 = p_i$, $r_i^0 = r_i$, $q_i^0 = q_i$ and $z_-^t = 1 - (\mathbf{p}^t, \mathbf{q}^t)$, $z_+^t = 1 + (\boldsymbol{\tau}^t, \mathbf{r}^t)$ stands for the normalising denominators where $\boldsymbol{\tau}^t = \frac{1}{2}(\mathbf{p}^t + \mathbf{q}^t)$.

For given starting vectors $\mathbf{p}, \mathbf{r}, \mathbf{q}$ the iteration of the mapping $*$ generates a discrete trajectory of the conflict dynamical system

$$\begin{Bmatrix} \mathbf{p}^t \\ \mathbf{r}^t \\ \mathbf{q}^t \end{Bmatrix} \xrightarrow{*,\,t} \begin{Bmatrix} \mathbf{p}^{t+1} \\ \mathbf{r}^{t+1} \\ \mathbf{q}^{t+1} \end{Bmatrix}, \qquad t \geq 0, \qquad (5)$$

where $\begin{Bmatrix} \mathbf{p}^t \\ \mathbf{r}^t \\ \mathbf{q}^t \end{Bmatrix}$ is called the state of the conflict dynamical system at the t-step of interaction. The problem is to study the behavior of these states as $t \to \infty$.

We consider that the interaction between $\mathbf{p}$ and $\mathbf{q}$ is repulsive, due to

$$|\mathbf{p}^t - \mathbf{r}^t| < |\mathbf{p}^{t+1} - \mathbf{p}^{t+1}| \text{ for all } t = 0, 1, \dots .$$

At the same time, $\mathbf{r}$ is attractive to one of the vectors $\mathbf{p}$ or $\mathbf{q}$, that is the distance between $\mathbf{r}$ and $\mathbf{p}$ or $\mathbf{q}$ decreases. So, interaction $*$ is *attractive and repulsive* simultaneous.

III. DESCRIPTION OF THE LIMITING STATES

Theorem 1. ([1, 6]) *For each fixed* $\mathbf{r}$ *and any couple of starting non-orthogonal stochastic vectors* $\mathbf{p}$, $\mathbf{q}$, *whose coordinates are changed using (4), there exist limits*

$$\lim_{t\to\infty} \mathbf{p}^t = \mathbf{p}^\infty, \qquad \lim_{t\to\infty} \mathbf{q}^t = \mathbf{q}^\infty.$$

Moreover,

$$\mathbf{p}^\infty \perp \mathbf{q}^\infty, \textit{ if } \ \boldsymbol{p} \neq \boldsymbol{q},$$

$$\mathbf{p}^\infty = \mathbf{q}^\infty, \textit{ otherwise.}$$

Let $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n_+$, $\boldsymbol{p} \neq \boldsymbol{q}$, $(\boldsymbol{p}, \boldsymbol{q}) > \mathbf{0}$, define

$$D_+ := \sum_{i \in \mathbb{N}_+} d_i, \qquad D_- := \sum_{i \in \mathbb{N}_-} d_i,$$

where $d_i = p_i - r_i$, $\mathbb{N}_+ := \{i: d_i > 0\}$, $\mathbb{N}_- := \{i: d_i < 0\}$.

Obviously $0 < D_+ = -D_- \leq 1$.

Theorem 2. ([1, 6]) *The coordinates of the limiting vectors* $\mathbf{p}^\infty, \mathbf{q}^\infty$ *have the following explicit representations*

$$p_i^\infty = \begin{cases} \dfrac{d_i}{D}, & i \in \mathbb{N}_+ \\ 0, & otherwise, \end{cases} \qquad q_i^\infty = \begin{cases} -\dfrac{d_i}{D}, & i \in \mathbb{N}_- \\ 0, & otherwise, \end{cases}$$

where $D = D_+ = D_-$.

Theorem 3. *For (5) in the case* $n > 2$ *for starting stochastic vectors* $\mathbf{r}$, *whose coordinates are changed using (4), there exist limits*

$$\lim_{t\to\infty} \mathbf{r}^t = \mathbf{r}^\infty.$$

Moreover

$$\mathbf{r}^\infty \perp \mathbf{p}^\infty \quad or \quad \mathbf{r}^\infty \perp \mathbf{q}^\infty.$$

Proposition 2. *Let* $n = 2$. *Then limit vectors coincide with one of the states:*

$$\mathbf{p}^\infty = (0; 1), (1; 0), \left(\frac{1}{2}; \frac{1}{2}\right),$$

$$\mathbf{q}^\infty = (1; 0), (0; 1), \left(\frac{1}{2}; \frac{1}{2}\right),$$

$$\mathbf{r}^\infty = (a; b), \qquad a + b = 1.$$

Theorem 4. *If there exist only one* i, *such that*

$$d_i = \max_{k \in \mathbb{N}_+, j \in \mathbb{N}_-} \{d_k, -d_j\},$$

then limiting coordinate of $\mathbf{r}^t$

$$r_i^\infty = 1 \ (r_k^\infty = 0 \ for \ all \ k \neq i).$$

Remark 1. Theorem 1-4 is valid not only for $\tau_i^t = \frac{p_i^t + q_i^t}{2}$, but also for τ_i^t, whose value depends on the three coordinates p_i^t, r_i^t, q_i^t, for example $\tau_i^t = \frac{1}{3}(p_i^t + r_i^t + q_i^t)$, $\tau_i^t = \min_i(p_i^t, r_i^t, q_i^t)$, $\tau_i^t = \max_i(p_i^t, r_i^t, q_i^t)$, etc.

IV. EXEMPLES

We obtain computer simulation for our model. Figure 1 with the initial state $\{\mathbf{p}, \mathbf{r}, \mathbf{q}\}$ and the limite state $\{\mathbf{p}^\infty, \mathbf{r}^\infty, \mathbf{q}^\infty\}$ illustrate the implementation of Theorem 3. The examples in Figures 2-4 demonstrate the assertion of Theorems 1, 2, 4 for a particular model of dynamical system.

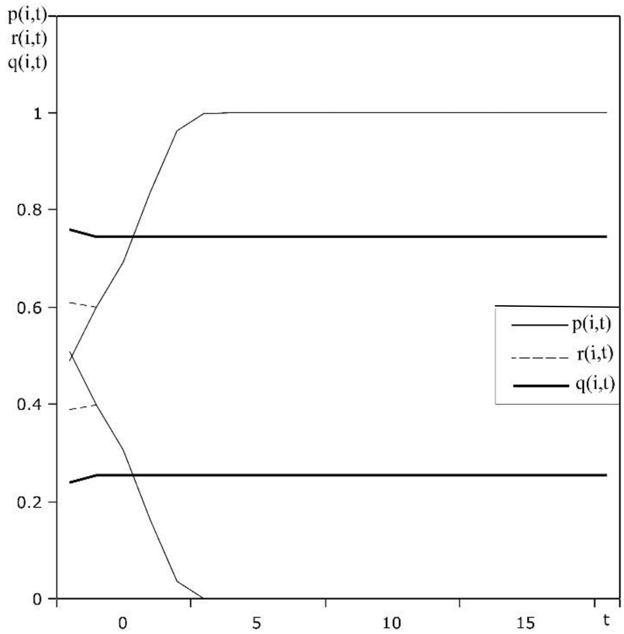

Figure 1. $n = 2$, $\mathbf{p} = (0{,}49;\ 0{,}51)$, $\mathbf{r} = (0{,}76;\ 0{,}24)$, $\mathbf{q} = (0{,}39;\ 0{,}61)$, $\mathbf{p}^\infty = (1;\ 0)$, $\mathbf{r}^\infty = (0{,}745;\ 0{,}255)$, $\mathbf{q}^\infty = (0;\ 1)$

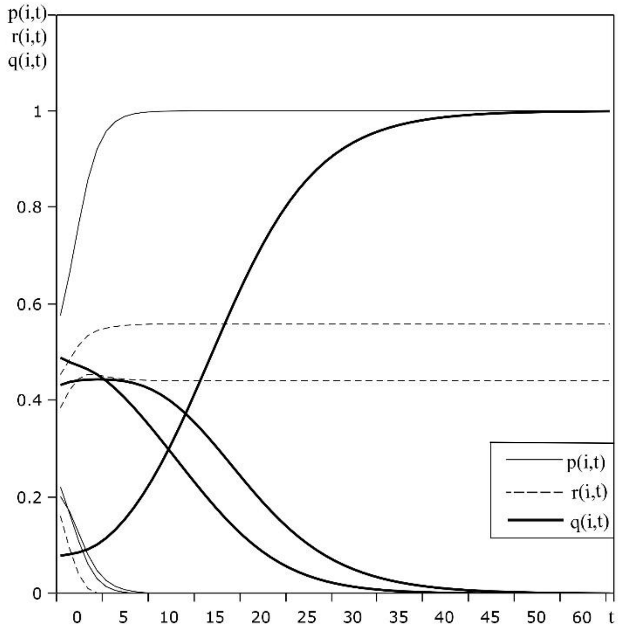

Figure 2. $n = 3$, $\mathbf{p} = (0{,}20187;\ 0{,}22175;\ 0{,}57638)$, $\mathbf{r} = (0{,}48895;\ 0{,}43225;\ 0{,}0788)$, $\mathbf{q} = (0{,}38499;\ 0{,}45388;\ 0{,}16113)$, $\mathbf{p}^{\infty} = (0;\ 0;\ 1)$, $\mathbf{r}^{\infty} = (0;\ 0;\ 1)$, $\mathbf{q}^{\infty} = (0{,}44;\ 0{,}56;\ 0)$

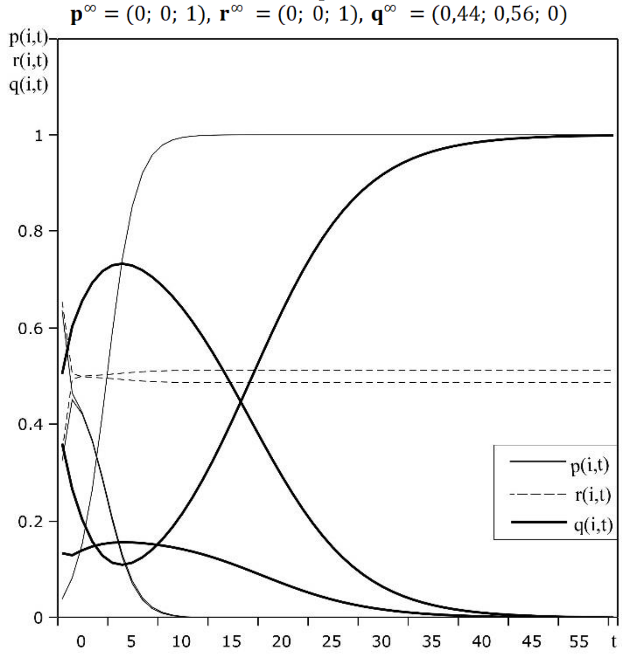

Figure 3. $n = 3$, $\mathbf{p} = (0{,}636;\ 0{,}325;\ 0{,}039)$, $\mathbf{r} = (0{,}505;\ 0{,}134;\ 0{,}361)$, $\mathbf{q} = (0{,}507;\ 0{,}257;\ 0{,}236)$, $\mathbf{p}^{\infty} = (0{,}655;\ 0{,}345;\ 0)$, $\mathbf{r}^{\infty} = (0;\ 0;\ 1)$, $\mathbf{q}^{\infty} = (0;\ 0;\ 1)$

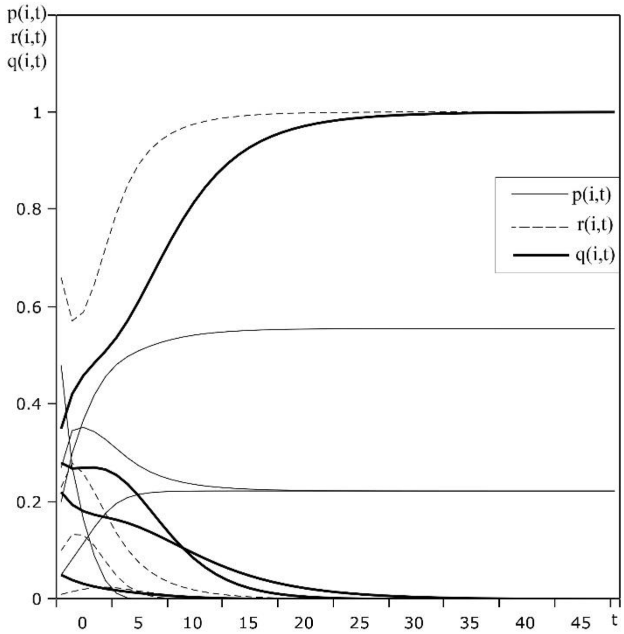

Figure 4. $n = 4$, $\mathbf{p} = (0{,}27;\ 0{,}2;\ 0{,}48;\ 0{,}05)$, $\mathbf{r} = (0{,}28;\ 0{,}22;\ 0{,}35;\ 0{,}05)$, $\mathbf{q} = (0{,}23;\ 0{,}1;\ 0{,}66;\ 0{,}01)$, $\mathbf{p}^{\infty} = (0{,}22;\ 0{,}55;\ 0\ ;\ 0{,}23)$, $\mathbf{r}^{\infty} = (0;\ 0;\ 1; 0)$, $\mathbf{q}^{\infty} = (0;\ 0;\ 1; 0)$.

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