Spaced Perturbations of "Predator-Prey" Dynamic Systems

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Abstract— **A mathematical model of "predator-prey" population interactions with regard to diffusive and migratory factors of individuals' dispersal in space is considered. In case the diffusive and migratory constituents are small as compared to other constituents of the process, it is suggested to search for the solution of a singular perturbed model problem in terms of asymptotical series.**

Keywords—"predator-prey" model; singular perturbed dynamic systems; asymptotical series

The Lotka-Volterra mathematical model of populations interactions [1,2] (or a "predator-prey" model) was propounded in the first half of the XX[th] century. In spite of further evolution, development of basic models modifications, and the extensive analysis of relevant systems, the model remains an important object for further investigations and the basis for elaborating new models of interaction processes and species competition within various domains.

Being a system of two ordinary differential equations of the first order, the Lotka-Volterra's classical model [1,2] is determined by the following assumptions: if there are no predators, preys reproduce themselves indefinitely; if there are no preys, predators become extinct. A detailed analysis for the solution of the model may be found in [3].

It should be pointed out that the obtained classical model-based dynamics of predators and preys' population developments reveal fluctuating features; moreover, in the absence of external factors the populations repeat fluctuation cycles indefinitely long, which might contradict actual observations to some extent.

The necessity for bringing the theoretical outcomes into proximity with the actual observations have provided for elaborating various methods of the classical model modification, e.g. by imposing restrictions on the increase of "prey" and "predator" populations as a result of intraspecific competition [4]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha u - \beta vu - \mu u^2, \\ \dfrac{\partial v}{\partial t} = -\gamma v + \delta vu - \rho v^2, \end{cases} \tag{1}$$

where α – coefficient of preys increase; β – coefficient of preys' losses when meeting predators; γ – coefficient of predators death; δ – coefficient of predators increase; μ, ρ – coefficients of prey and predator intraspecific competition; *t* – time.

One may obtain realistic results by using the models, which embed the time of argument lateness [5], take account of parameters' random fluctuations [6] and random external effects for describing factual stochastic processes, occurring in the system [7], etc.

Relative simplicity, universality and sufficient non-linearity of the Lotka-Volterra model contributed to the popularization and extensive use of its modifications for modeling and studying various interaction processes. The "prey-predator" system is rather popular at modeling the interaction processes in biology [3,4,8,10], in particular, in immunology [12,13,14], ecology [5,9,11]. The similar models are also widely used in economics [15,16,17].

It should be pointed out that the observation of populations' behavior in a real-world environment indicates the existence of such natural phenomena as concentration or diffusion of the populations' individuals. That is why, a more realistic modeling of species interaction processes requires to take into account some spatial factors, in particular, diffusion and migration, which substantially complicate the search and analysis for the solution of such model problems. However, when diffusive and migratory constituents of dispersal in space are small as compared to other constituents of species increase and interaction processes, the relevant model problem may be represented in the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha u - \beta v u - \mu u^2 - \varepsilon V_* \dfrac{\partial u}{\partial x} + \varepsilon^2 D_* \dfrac{\partial^2 u}{\partial x^2}, \\ \dfrac{\partial v}{\partial t} = -\gamma v + \delta v u - \rho v^2 - \varepsilon V^* \dfrac{\partial v}{\partial x} + \varepsilon^2 D^* \dfrac{\partial^2 v}{\partial x^2}, \end{cases} \tag{2}$$

at $0 < x < l,\ t > 0$,

under initial conditions

$$u(x,0) = u_*(x), \quad v(x,0) = v_*(x), \quad 0 \le x \le l, \tag{3}$$

and boundary conditions

$$u'_x(0,t) = u'_x(l,t) = v'_x(0,t) = v'_x(l,t) = 0,\ 0 \le t \le \overline{T} < \infty, \tag{4}$$

where x – spatial coordinate; t – time; $u(x,t)$, $v(x,t)$ – density of predators and preys accordingly; $\varepsilon^2 D_*$, $\varepsilon^2 D^*$ – “diffusion” coefficients that characterize the intensity of spatial distribution of prey and predator population individuals accordingly; ε – small parameter; εV_*, εV^* – migration velocity of prey and predator population individuals accordingly ($\varepsilon V_*, \varepsilon V^* > 0$).

The solution on a singularly perturbed system (2) may be found in [18,19,20]:

$$\begin{aligned} u &= \sum_{i=o}^{n} \varepsilon^i u_i(x,t) + \sum_{i=o}^{n} \varepsilon^i \underset{\sim}{u}_i(\underset{\sim}{x},t) + \sum_{i=o}^{n} \varepsilon^i \tilde{u}_i(\tilde{x},t) + R_{*n}, \\ v &= \sum_{i=o}^{n} \varepsilon^i v_i(x,t) + \sum_{i=o}^{n} \varepsilon^i \underset{\sim}{v}_i(\underset{\sim}{x},t) + \sum_{i=o}^{n} \varepsilon^i \tilde{v}_i(\tilde{x},t) + R_n^*, \end{aligned} \tag{5}$$

where $R_{*n} = O(\varepsilon^{n+1}),\ R_n^* = O(\varepsilon^{n+1})$ – remainders; $u_i(x,t),\ v_i(x,t)$ – elements of regular parts of the asymptotics; $\underset{\sim}{u}_i(\underset{\sim}{x},t)$, $\tilde{u}_i(\tilde{x},t)$, $\underset{\sim}{v}_i(\underset{\sim}{x},t)$, $\tilde{v}_i(\tilde{x},t)$ – corrections near $x=0$, $x=l$; $\underset{\sim}{x} = \dfrac{x}{\varepsilon^\lambda}$, $\tilde{x} = \dfrac{l-x}{\varepsilon^\lambda}$ – extension variables [18,19,20] with the corresponding index λ.

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