The Problem of Two Bodies with a Finite Velocity of Gravity

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Abstract – **From the moment of Newton's discovery of the law of universal gravitation, ordinary differential equations were used to study the motion of bodies, since it was assumed that the velocity of gravitation is infinite. However, in reality the velocity of gravity is finite, which is consistent with the theory of relativity of Einstein, which postulated that the velocity of gravity matches the velocity of light, and the studies conducted by S. Kopeikin and E. Fomalont on the fundamental limit of the velocity of gravity. Due to the delay of the gravitational field for studying the motion of bodies, the mathematical apparatus based on differential equations with a delay argument is the most acceptable one. These equations are used to construct and study the mathematical model of the motion of two bodies. It is shown that the motion of these bodies with identical masses (with finite velocity of gravity!) is not carried out in accordance with Kepler's laws.**

Keywords: **two-body problem, two-body problem with finite gravity velocity.**

I. INTRODUCTION

The article is devoted to the application of differential equations with a delayed argument to the construction and study of a mathematical model of the motion of two bodies under the action of the forces of the law of gravity, taking into account the finite speed of gravity.

The application to the study of the motion of two bodies of these equations is natural, since the gravitational effect of one body on another cannot be instantaneous and it takes time during which the gravitational field passes the distance between these bodies.

II. TWO-BODY PROBLEM IN CLASSICAL CELESTIAL MECHANICS

Since the discovery of the law of universal gravitation by I. Newton (1643-1727) in 1687, ordinary differential equations were used to study the motion of bodies, since it was believed that the speed of gravity is infinite.

The simplest task of classical celestial mechanics is the two-body problem. According to Newton's second law and the law of universal gravitation, the differential equations of motion of the bodies of this problem in some fixed Cartesian coordinate system have the form

$$\begin{cases} \ddot{\vec{r}}_1(t) = -\dfrac{Gm_2}{|\vec{r}_2(t) - \vec{r}_1(t)|^3}\,(\vec{r}_1(t) - \vec{r}_2(t)), \\ \ddot{\vec{r}}_2(t) = -\dfrac{Gm_1}{|\vec{r}_2(t) - \vec{r}_1(t)|^3}\,(\vec{r}_2(t) - \vec{r}_1(t)), \end{cases} \quad (1)$$

where G is the gravitational constant, m_1 and m_1 are the masses of bodies, and $|\vec{r}_2(t) - \vec{r}_1(t)|$ is the Euclidean length of the vector $\vec{r}_2(t) - \vec{r}_1(t)$.

The study of system (1) is given in many works (see, for example, [1]–[3]).

I. Newton found a general solution to system (1) and this solution is given a geometric shape. The trajectories of the motion of one body relative to another and relative to the center of mass are finite sections.

III. A MATHEMATICAL MODEL OF THE MOTION OF TWO BODIES, TAKING INTO ACCOUNT THE FINITE SPEED OF GRAVITY

In real space, the speed of gravity cannot be infinite, as in Newton's theory. This statement is consistent with Einstein's theory of relativity, in which it is postulated that the speed of gravity coincides with the speed of light c, and with the studies of Kopeikin S.M. and Fomalont E.B. on the fundamental limit of the speed of gravity [4]. Application of this property of gravity allows you to build a model of motion of two bodies, which uses not ordinary differential equations, but differential equations with a delayed argument, and find new properties of the motion of these bodies.

Consider the movement of points M_1 and M_2 with masses m_1 and m_2 in a rectangular coordinate system x, y, z with the origin at point O. We consider the coordinate system to be inertial. The position of points M_1 and M_2 at time t is determined by their vectors $\vec{r}_i(t)$, $i = \overline{1,2}$.

If the speed of gravity were infinite, as in classical celestial mechanics, then, on the basis of the law of universal gravitation at time t, point M_2 pulled point M_1 with the power of

$$\vec{F}(t) = \frac{Gm_1m_2}{|\vec{r_2}(t) - \vec{r_1}(t)|^3}(\vec{r_2}(t) - \vec{r_1}(t)). \qquad (2)$$

However, due to the finite speed of gravity, another force acts on point M_1

$$\vec{F_1}(t) = \frac{Gm_1m_2}{|\vec{r_2}(t - \tau_2(t)) - \vec{r_1}(t)|^3}(\vec{r_2}(t - \tau_2(t)) - \vec{r_1}(t)). \quad (3)$$

The delay of gravity $\tau_2(t)$ in (3) is determined by the equality

$$c\tau_2(t) = |\vec{r_2}(t - \tau_2(t)) - \vec{r_1}(t)|, \qquad (4)$$

where c is the speed of gravity.

Indeed, let points M_2 and M_1 move along the corresponding trajectories, parts of which are shown in Fig. 1, with speeds $\vec{v_2}(t) = \dot{\vec{r_2}}(t)$ and $\vec{v_1}(t) = \dot{\vec{r_1}}(t)$ and at time $t - \tau_2(t)$, where $\tau_2(t)$ satisfies (4), are at points C and D, respectively. Over a period of time $[t - \tau_2(t), t]$, point M_2 will move from point C to point A, and point M_1 will move from point D to point B. This period of time is enough for the gravitational field to propagate from speed c from point C to point B. Therefore, at time t, point B is not affected by force (2), but by force (3).

Similarly, based on the final gravity velocity, a force acts on point M_2

$$\vec{F_2}(t) = \frac{Gm_1m_2}{|\vec{r_1}(t - \tau_1(t)) - \vec{r_2}(t)|^3}(\vec{r_1}(t - \tau_1(t)) - \vec{r_2}(t)). \quad (5)$$

The delay of gravity $\tau_1(t)$ to (5) is determined by the equality

$$c\tau_1(t) = |\vec{r_1}(t - \tau_1(t)) - \vec{r_2}(t)|. \qquad (6)$$
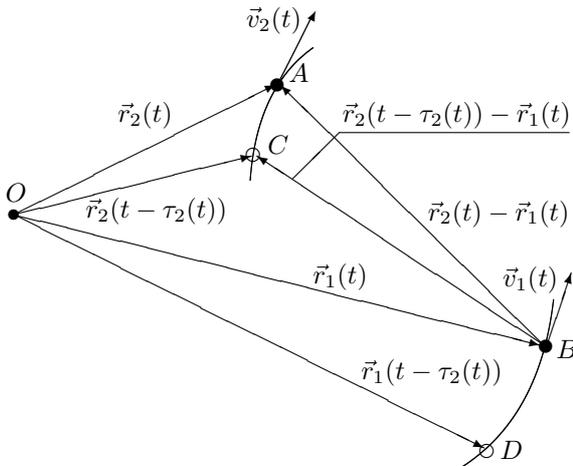
Fig. 1. Points M_1 and M_2 at time points $t - \tau_2(t)$ and t.

By virtue of implicit function theorems [5, pp. 449–453], functions $\tau_2(t)$ and $\tau_1(t)$ are continuous and differentiable.

From the above reflections, taking into account Newton's second law and the universal law, it follows that the motion of two points M_1 and M_2 with masses m_1 and m_2 is described by a system of equations

$$\begin{cases} m_1\ddot{\vec{r_1}}(t) = \vec{F_1}(t), \\ m_2\ddot{\vec{r_2}}(t) = \vec{F_2}(t). \end{cases}$$

From here, (3) and (5) we get a system of equations that describes the motion of points M_1 and M_2. This system has the form

$$\begin{cases} \ddot{\vec{r_1}}(t) = \dfrac{Gm_2}{|\vec{r_2}(t - \tau_2(t)) - \vec{r_1}(t)|^3}(\vec{r_2}(t - \tau_2(t)) - \vec{r_1}(t)), \\[2mm] \ddot{\vec{r_2}}(t) = \dfrac{Gm_1}{|\vec{r_1}(t - \tau_1(t)) - \vec{r_2}(t)|^3}(\vec{r_1}(t - \tau_1(t)) - \vec{r_2}(t)). \end{cases}$$
$$(7)$$

Note that system (7) is a system of differential equations with a retarded argument and differs significantly from system (1).

Elements of the theory of differential equations with delayed argument are presented in [6]–[9].

For a complete description of the motion of points M_1 and M_2, it is necessary for system (7) to use additional initial or boundary conditions (see problems 1 and 2).

PROBLEM 1. We fix arbitrary time moment t_0 and continuous on segments $[t_0 - \tau_i(t_0), t_0]$, $i = \overline{1,2}$, vector functions $\vec{\varphi}_{0,i}(s)$ and $\vec{\varphi}_{1,i}(s)$, $i = \overline{1,2}$, respectively. It is necessary to find solutions $\vec{r_i}(t)$, $i = \overline{1,2}$, system (7), which satisfy the initial conditions

$$\begin{cases} \vec{r_i}(s) = \vec{\varphi}_{0,i}(s), & s \in [t_0 - \tau_i(t_0), t_0], \\ \dot{\vec{r_i}}(s) = \vec{\varphi}_{1,i}(s), & s \in [t_0 - \tau_i(t_0), t_0], \quad i = \overline{1,2}. \end{cases} \quad (8)$$

PROBLEM 2. Let t_1 and t_2 be arbitrary instants of time, for which $t_1 < t_2 - \tau_i(t_2)$, $i = \overline{1,2}$. Consider twice continuously differentiated on segments $[t_1 - \tau_i(t_1), t_1]$ and $[t_2 - \tau_i(t_2), t_2]$, $i = \overline{1,2}$, vector functions $\vec{\psi}_{1,i}(s)$ and $\vec{\psi}_{2,i}(s)$, $i = \overline{1,2}$, respectively. It is necessary to find solutions $\vec{r_i}(t)$, $i = \overline{1,2}$, system (7) that satisfy the conditions

$$\begin{cases} \vec{r_i}(s_1) = \vec{\psi}_{1,i}(s_1), & s_1 \in [t_1 - \tau_i(t_1), t_1], \\ \vec{r_i}(s_2) = \vec{\psi}_{2,i}(s_2), & s_2 \in [t_2 - \tau_i(t_2), t_2], \quad i = \overline{1,2}. \end{cases} \quad (9)$$

Equations (4), (6) and (7) with conditions (8) or (9) are mathematical models of the motion of two bodies, which take into account the speed of gravity.

IV. KERLER'S LAWS AND THE MAIN PURPOSE OF THE ARTICLE

The movement of bodies (in particular, the movement of planets) in classical celestial mechanics is carried out according to the three laws of Kepler (1571-1630), discovered in 1605, 1601 and 1618, respectively [10, pp. 261, 262]:

1. The trajectories of the planets are ellipses, in one of the foci of which (common to all planets) is the Sun.

2. The areas that are described by the radius vectors of the planets drawn from the Sun increase in proportion to time.

3. The squares of the periods of rotation of the planets around the Sun are correlated as cubes of the major semiaxes of their orbits.

These laws in celestial mechanics (with a finite speed of gravity) have some inaccuracies.

Indeed, based on the delay of the gravitational field when the planets move around the Sun, the attracting point for each planet at a moment in time t is not the center of the Sun, as in Kepler's first law, but another point in space in which the center of the Sun was at some moment in time $t - \tau(t)$, where $\tau(t)$ is a delay depending on the location of the planet and the Sun, similar to the delays $\tau_2(t)$ and $\tau_1(t)$, which satisfy (4) and (6).

Simple calculations show that each planet rotates around its "own" attracting point, which does not coincide with the center of the Sun, and these points do not coincide in pairs.

Therefore, due to the displacement of the attracting points, the motion of the bodies does not agree with Kepler's first law (with respect to attracting points).

It does not follow from this conclusion that ellipses cannot be trajectories of bodies.

The purpose of the article is to show that, based on the finite speed of gravity, the motion of bodies of the same mass cannot follow Kepler's laws.

V. The impossibility of moving bodies along an ellipse behind Kepler's second law

Consider the system of equations (7), which describes the motion of two points M_1 and M_2 with masses m_1 and m_2, taking into account the finite speed of gravity. Due to the presence of delays $\tau_1(t)$ and $\tau_2(t)$ in the equations, which is not in system (1), it is impossible to find a general solution to this system. However, with additional restrictions on the equations of the system, important properties of the motion of points can be obtained.

In the future, we assume that the masses of points M_1 and M_2 are the same and equal to m.

Suppose that points M_1 and M_2 move along the ellipse and beyond Kepler's second law.

Then the movement of point M_1 relative to point M_2 will be periodic.

We use the equations of motion of point M_1 relative to point M_2, i.e. equations for the vector function

$$\vec{r}(t) = \vec{r}_1(t) - \vec{r}_2(t).$$

Since the points M_1 and M_2 are affected only by the forces arising from the interaction of these points, and these forces are equal in magnitude and opposite in direction, the accelerations $\ddot{\vec{r}}_1(t)$ and $\ddot{\vec{r}}_2(t)$ of the movement of the points have similar properties, i.e.

$$\ddot{\vec{r}}_1(t) + \ddot{\vec{r}}_2(t) \equiv 0$$

and therefore

$$\dot{\vec{r}}_1(t) + \dot{\vec{r}}_2(t) \equiv \vec{v}, \qquad (10)$$

where $\vec{v}$ is a constant vector. A ratio of (10) means that the center of the masses of points M_1 and M_2, which is determined by the vector function $\vec{r}_O(t) = \frac{1}{2}(\vec{r}_1(t) + \vec{r}_2(t))$, moves in a straight line at a constant speed.

Without loss of generality, we can assume that the origin of the inertial coordinate system coincides with the center of mass of the points M_1 and M_2. Then the ratio of (10) is given in the form

$$\dot{\vec{r}}_1(t) + \dot{\vec{r}}_2(t) \equiv 0.$$

Then $\vec{v} = 0$ and, therefore, the speeds $\vec{v}_1(t) = \dot{\vec{r}}_1(t)$ and $\vec{v}_2(t) = \dot{\vec{r}}_2(t)$ of the movement of points M_1 and M_2 are opposite.

It follows from the above reasoning that points M_1 and M_2 are located on an ellipse with major axis $A_1 A_2$ symmetrically with respect to its center O (see Fig. 2).

The attracting points M_1^* and M_2^* for points M_2 and M_1, respectively, at time t are also placed on the ellipse symmetrically with respect to its center O.
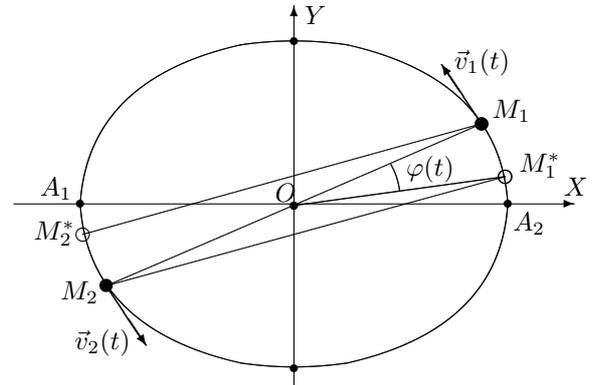
Fig. 2. Location of points M_1, M_2, M_1^* and M_2^*.

The coordinates of points M_1^*, M_2^*, M_1 and M_2 at time t are determined by vectors $\vec{r}_1(t - \tau_1(t))$, $\vec{r}_2(t - \tau_2(t))$, $\vec{r}_1(t)$ and $\vec{r}_2(t)$, respectively, and, obviously, according to Fig. 2 are carried out equalities

$$\left| \overrightarrow{M_1 M_2^*} \right| = \left| \overrightarrow{M_2 M_1^*} \right| =$$

$$= |\vec{r}_1(t) - \vec{r}_2(t - \tau_2(t))| = |\vec{r}_2(t) - \vec{r}_1(t - \tau_1(t))|.$$

Denote by $d(t)$ the length of the vector $\overrightarrow{M_1 M_2^*}$.

Then, in system (7), delays $\tau_1(t)$ and $\tau_2(t)$ are the same (we denote them by $\tau(t)$), and according to the equations of this system, function $\vec{r}(t)$, which describes the motion of point M_1 relative to point M_2, is a solution to equation

$$\ddot{\vec{r}}(t) = -\frac{Gm}{d^3(t)} \left(\vec{r}(t) + \vec{r}(t - \tau(t)) \right). \tag{11}$$

Next, we use a rectangular Cartesian coordinate system with the origin at the center of ellipse O. Let axis Ox of this coordinate system pass through the vertices A_1 and A_2 of the ellipse, and axis Oy be perpendicular to axis Ox and pass through point O.

Then function $\vec{r}(t)$ can be given in the form

$$\vec{r}(t) = 2(x(t), y(t)), \tag{12}$$

where $x(t)$ and $y(t)$ are the coordinates of point M_1 at time t and according to (11)

$$\begin{cases} \ddot{x}(t) = -\dfrac{Gm}{d^3(t)} \left(x(t - \tau(t)) + x(t) \right), \\ \ddot{y}(t) = -\dfrac{Gm}{d^3(t)} \left(y(t - \tau(t)) + y(t) \right). \end{cases} \tag{13}$$

We pay attention to one important property of the solutions of the system of equations (13) using functions (12),

$$\vec{v}(t) = \dot{\vec{r}}(t) = 2(\dot{x}(t), \dot{y}(t))$$

and

$$v_\sigma(t) = 2 \begin{vmatrix} x(t) & y(t) \\ \dot{x}(t) & \dot{y}(t) \end{vmatrix}. \tag{14}$$

In the case of classical celestial mechanics $v_\sigma(t)$ is a sectoral velocity of point M_1 relative to point M_2 and

$$v_\sigma(t) \equiv v, \tag{15}$$

where v is a constant [11, p. 134].

Sectoral speed $v_\sigma(t)$ of the movement of point M_1 relative to point M_2, if we take into account the finiteness of the speed of gravity, is not a constant value, but strictly increases.

Indeed, given (13) and (14)

$$\frac{1}{2} \frac{dv_\sigma(t)}{dt} = \frac{d}{dt} \begin{vmatrix} x(t) & y(t) \\ \dot{x}(t) & \dot{y}(t) \end{vmatrix} = \begin{vmatrix} x(t) & y(t) \\ \ddot{x}(t) & \ddot{y}(t) \end{vmatrix} =$$

$$-\frac{Gm}{d^3(t)} \begin{vmatrix} x(t) & y(t) \\ x(t - \tau(t)) + x(t) & y(t - \tau(t)) + y(t) \end{vmatrix} =$$

$$= -\frac{Gm}{d^3(t)} \begin{vmatrix} x(t) & y(t) \\ x(t - \tau(t)) & y(t - \tau(t)) \end{vmatrix}.$$

From these equalities, in particular, we obtain the identity (15) when $\tau(t) \equiv 0$.

Let t_0 be an arbitrary moment of time from the interval during which the movement of points M_1 and M_2 occurs. Then for all $t \geqslant t_0$

$$\frac{d}{dt} \begin{vmatrix} x(t) & y(t) \\ \dot{x}(t) & \dot{y}(t) \end{vmatrix} = \frac{Gm}{d^3(t)} \Delta(t), \tag{16}$$

where

$$\Delta(t) = \begin{vmatrix} x(t - \tau(t)) & y(t - \tau(t)) \\ x(t) & y(t) \end{vmatrix}.$$

We use Fig. 3, which shows the vectors $\vec{r}(t)$, $\vec{v}(t)$, $\vec{r}(t - \tau(t))$ and part of the trajectory the motion of the point M_1 relative to the point M_2. The point M_3 in this figure is the end of the vector $\vec{v}(t)$.
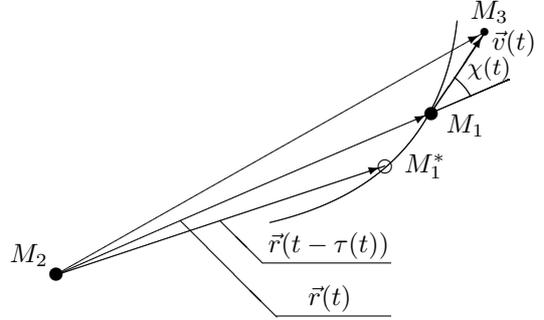
Fig. 3. To the growth of sectoral speed.

Hence the geometric meaning of the second-order determinant implies that the area $S_{M_1 M_2 M_3}(t)$ of triangle $M_1 M_2 M_3$ is a strictly increasing function and the rate of change of this area is proportional to the area $S_{M_1 M_2 M_1^*}(t)$ of triangle $M_1 M_2 M_1^*$, i.e.

$$\frac{dS_{M_1 M_2 M_3}(t)}{dt} = \frac{Gm}{d^3(t)} S_{M_1 M_2 M_1^*}(t), \quad t \geqslant t_0.$$

The obtained property of the solutions of the system of equations (13) indicates an increase in the sectoral velocity of motion of point M_1 relative to point M_2 (the law of increasing the sectoral velocity).

An increase in this velocity contradicts the periodicity of the motion of point M_1 relative to point M_2.

Therefore, the assumption that points M_1 and M_2 move along the ellipse and under Kepler's second law is false.

Thus, according to Section IV, the movement of the point M_1 does not occur around the point M_2, but around the point of attraction M_2^*, and the first Kepler's law with respect to the point of attraction does not hold. Also, the sectoral velocity of movement of point M_1 relative to point M_2 is not constant but strictly increasing, and therefore the second Kepler law is not fulfilled.

To show that the third Kepler law is not fulfilled, it suffices to show that the trajectory of motion of point M_1

relative to point M_2 is not an ellipse. Then, for points M_1 and M_2, the first Kepler law will not be fulfilled not only with respect to the attraction points, but also with respect to the motions of their motions.

VI. The impossibility of moving bodies along an ellipse

Suppose that points M_1 and M_2 move along the ellipse, as in Section V.

According on the increase in sector velocity $v_\sigma(t)$ and the movement of points M_1 and M_2 by ellipse for some numbers $\delta_1 > 0$ and $\delta_2 > 0$ for velocity $\vec{v}(t)$ of movement of point M_1 relative to point M_2 and for the angle $\chi(t)$ between vectors $\overline{r}(t)$ and $\vec{v}(t)$, the inequalities hold

$$|\vec{v}(t)| \geqslant \delta_1, \quad t \geqslant t_0,$$

and

$$\delta_2 \leqslant \chi(t) \leqslant \pi - \delta_2, \quad t \geqslant t_0.$$

Therefore, for some number $\delta_3 > 0$, the inequalities

$$d(M_1, M_1^*) \geqslant \delta_3, \quad t \geqslant t_0, \tag{17}$$

will be satisfied for the distance $d(M_1, M_1^*)$ between points M_1 and M_1^*.

By virtue of (17) and the assumption that the points M_1 and M_1^* move along the ellipse for a certain number $\delta_4 > 0$, the angle $\varphi(t)$ between vectors $\vec{r}_1(t) = \overrightarrow{OM_1}$ and $\vec{r}_1(t - \tau(t)) = \overrightarrow{OM_1^*}$ (see Fig. 2) for each $t \geqslant t_0$ satisfies the inequality

$$\frac{\pi}{2} > \varphi(t) \geqslant \delta_4.$$

Considering that the area $S(t)$ of the parallelogram constructed on $\vec{r}_1(t)$ and $\vec{r}_1(t - \tau(t))$ coincides with the value of the determinant $\Delta(t)$ and $S(t) = |\vec{r}_1(t)| \cdot |\vec{r}_1(t - \tau(t))| \sin \varphi(t)$ (see Fig. 2), we obtain that for some number $\delta_5 > 0$

$$\Delta(t) \geqslant \delta_5, \quad t \geqslant t_0. \tag{18}$$

Since $0 < 2a \leqslant d(t) \leqslant 2b$, $t \geqslant t_0$, where a and b are the lengths of the minor and major semi-axes of the ellipse (see Fig. 2), it is based on (16) and (18),

$$\lim_{t \to +\infty} v_\sigma(t) = +\infty,$$

which is impossible, since the speed cannot be greater than c.

Therefore, the assumption that points M_1 and M_2 move along an ellipse is false.

Thus, the motion of two bodies of the same mass in real space with a finite gravity velocity does not satisfy Kepler's laws.

VII. Additional remarks and literature

The general case of a system of n bodies, taking into account the finite speed of gravity, was considered by the author in article [12]. This article also pays attention to the construction of a mathematical model of the solar system with a finite speed of gravity and to obtain some new properties for it.

In [13], the problem of two bodies of arbitrary masses was studied taking into account the speed of gravity. The non-Kaplerian behavior and instability of the motion of these bodies are also shown.

In [14], systems of differential equations with delays and constraints on delays and derivatives of solutions, which are used in [12] and [13], are studied.

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