Application of a Differential Criterion to the Study of the Convergence of Number Series

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Abstract – **It is shown that most of the number series that are considered in the collection of problems of G. N. Berman in the course of mathematical analysis can be examined for convergence using the differential criterion V. Yu. Slyusarchuk.**

Keywords: **numerical series, differential sign of convergence.**

Consider the number series

$$\sum_{n=1}^{\infty} a_n, \tag{1}$$

where

$$a_n \geq a_{n+1} > 0$$

for all $n \geq 1$.

We assume that there exists a continuously differentiable and monotone on $[1, +\infty)$ function $f(x)$, for which

$$f(n) = a_n$$

for all $n \geq 1$.

To study the convergence of the number series (1), we use a differential feature based on the following statement obtained by V. Yu. Slyusarchuk [1].

Consider the functions

$$S_1(x) = -x\frac{d\ln f(x)}{dx},$$

$$S_2(x) = (S_1(x) - 1)\ln x,$$

$$S_3(x) = (S_2(x) - 1)\ln\ln x,$$

$$\vdots$$

$$S_{n+1}(x) = (S_n(x) - 1)\underbrace{\ln\ln\ldots\ln}_{n \text{ times}} x, \quad n > 1.$$

THEOREM [1]. If

$$\varliminf_{x\to+\infty} S_p(x) < 1$$

for some $p \in \mathbb{N}$, then the number series (1) converges. If

$$S_p(x) \leq 1$$

for some $p \in \mathbb{N}$ and all sufficiently large $x > 1$, then the number series (1) diverges.

With the help of this statement, we can investigate most of the number series for convergence, which are considered in the collection of problems of G. N. Berman in the course of mathematical analysis [2].

The effectiveness of the above theorem (see [1, p. 70]) is well illustrated by the generalized harmonic series

$$\sum_{n=1}^{\infty} n^{-p}, \quad p \in \mathbb{R}. \tag{2}$$

In the case of this series

$$f(x) = x^{-p}.$$

It's obvious that

$$S_1(x) = -x\frac{d\ln x^{-p}}{dx} = p.$$

According to the above theorem, series (2) converges only for $p > 1$.

To study convergence, for example, of number series

$$\sum_{n=3}^{\infty} n^{-1}(\ln n)^{-p}$$

and

$$\sum_{n=9}^{\infty} n^{-1}(\ln n)^{-1}(\ln\ln n)^{-p},$$

you need to use functions $S_2(x)$ and $S_3(x)$.

REFERENCES

[1] V. Yu. Slyusarchuk, "General theorems on the convergence of number series", Rivne: RSTU, 2003.

[2] G. N. Berman. "Collection of tasks on the course of mathematical analysis", Moscow: Nauka, 1985.