About Averaging in Hyperbolic Equation under the Influence of Multifrequency Disturbances

Yaroslav Bihun
Dept. of Applied Mathematics and IT
Yuriy Fedkovych Chernivtsi National University
Chernivtsi, Ukraine
yaroslav.bihun@gmail.com

Ihor Skutar
Dept. of Applied Mathematics and IT
Yuriy Fedkovych Chernivtsi National University
Chernivtsi, Ukraine
ihor27@gmail.com

Abstract—The research deals with the existence of the solution of the initial problem for hyperbolic equation under the multifrequency disturbances, which are described by the system of ordinary differential equations (ODE) with multipoint and integral conditions. The averaging method over fast variables is grounded and estimation of accuracy of the method which obviously depends on the small parameter was found.

Keywords—averaging method; multifrequency problem; system; resonance; small parameter; integral conditions.

Mathematical models in the problems of managing dynamic systems, in particular with delay, under the influence of multifrequency disturbances, string generators, and processes in chemical kinetics are described with the help of differential equations with ordinary and partial derivatives [1–3].

In this paper we consider the linear hyperbolic equation with delay

\[ u_{tt} = c^2 u_{xx} + \mu \sum_{v=1}^{s} b_v(x, \tau) u_{\delta v} + \mu f(x, \tau, a_A, \varphi_\Theta) \]

under the influence of multi-frequency disturbances, which are determined by the system of equations

\[ \frac{da}{d\tau} = X(\tau, a_A, \varphi_\Theta), \]
\[ \frac{d\varphi_\Theta}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y(\tau, a_A, \varphi_\Theta). \]

Here \( \Delta = \varepsilon t \in [0, L] \), \( x = \varepsilon z \in \mathbb{R}, \varepsilon \in (0, \varepsilon_0], \varepsilon_0 \ll 1, \mu = \varepsilon^2, a_A(\tau) = (a(\lambda_1 \tau), ..., a(\lambda_p \tau)), \varphi_\Theta(\tau) = (\varphi(\theta_1 \tau), ..., \varphi(\theta_q \tau)), u_A(x, \tau) = (u(x, \delta_1 \tau), ..., u(x, \delta_r \tau)), \lambda_i, \theta_j, \delta_k \in (0, 1). \)

Moving to the slow variables in the equation (1) we will get the equation

\[ u_{tt} = c^2 u_{xx} + \sum_{v=1}^{s} b_v(x, \tau) u_{\delta v} + f(x, \tau, a_A, \varphi_\Theta). \]

For the solutions of the system of equations the following conditions are given:

\[ u(x, 0) = \varphi(x), u_x(x, 0) = \psi(x), x \in \mathbb{R}. \]

For the solution of the system of equations under the influence of multifrequency disturbances, which are described by the system of ordinary differential equations (ODE) with multipoint and integral conditions. The averaging method over fast variables is grounded and estimation of accuracy of the method which obviously depends on the small parameter was found.

The main condition of the research of the multifrequency systems is the condition of passage of the system through resonance. For the system (1) the condition of resonance \( (k, \omega(\tau, a(\tau, \varepsilon))) \approx 0, z^m \ni k \neq 0, (\cdot, \cdot) \) is a scalar product. For the system with transformed arguments a condition is found [4]:

\[ \sum_{v=1}^{q} \theta_v(k_v, \omega(\theta_v \tau)) = 0, k_v \in \mathbb{R}^m, ||k|| \neq 0. \]

If \( V(\tau) \neq 0, \tau \in [0, L] \), where \( V(\tau) \) is Wronskian determinant by the system of functions \( \{\omega(\theta_1 \tau), ..., \omega(\theta_q \tau)\} \), then the system gets out of resonance.
For one-frequency system with one linearly transformed argument the Wronskian determinant is

\[ V(\tau) = \begin{vmatrix} \omega(\tau) \\
\frac{d\omega(\tau)}{d\tau} \\
\frac{d\omega(z)}{dz} \end{vmatrix} = \theta \tau, \theta \in (0,1). \]

The grounding of the averaging method is grounded on the estimation of the corresponding to the system (2) oscillation integral.

For the ordinary differential equations such an estimation is received in [5], for the systems with linearly transformed arguments used in [4, 6, 7]. The grounding of existence and uniqueness of the problem (2), (5), (6) is founded on the basis of sufficient conditions for existence of the solution of the initial problem. Whereby we receive the estimation of accuracy of the method, the order of which is \( e^a, a = (mq)^{-1}. \)

To prove the main statement we use the following integral inequality, which is synthesis of the result [8].

**Lemma.** Let the constant \( \alpha \geq 0 \), functions \( \alpha_p \in C(I, I), \alpha_p(t) \leq t \) for \( t \in I \), \( \alpha_p \in C(I, R_+), v = \int_1^n. \) If \( u \in C(I, R_+) \) and

\[ u(t) = a + \sum_{v=1}^{n} \int a_v(s)u(s)ds, \]

then for \( t \in I \)

\[ u(t) \leq a \cdot \exp \left( \sum_{v=1}^{n} \int a_v(s)ds \right). \]

**Corollary.** If \( t_0 = 0, \alpha_v(t) = \lambda_v t, \lambda_v \in (0, 1), \alpha_\text{p} \geq 0 \), then

\[ u(t) \leq a \cdot \exp \left( \sum \alpha_v \lambda_v t \right). \]

**Theorem.** Suppose, that conditions are true:
1) vector-functions \( X, Y, f_i, g_i, i = 1, 2, \nu = 1, q \) and function \( f \) are differentiable over the variables \( x, t, \alpha_\text{p} \) and \( m+1 \) once differentiable over the fast variables \( \varphi_k \);
2) \( \alpha_p \in C^{m+1}[0, L], v = \int_1^n, \) and Wronskian determinant by the system of functions \( \{\omega(\theta_1 \tau), ..., \omega(\theta_q \tau)\} \) is not equal to zero on \([0, L] \};
3) \( \varphi \in C^2(\mathbb{R}), \psi \in C^4(\mathbb{R}); \)
4) functions \( b_v \in C(\mathbb{R} \times (0,1)), v = \int_1^n, \) there is a unique solution for the averaged problem, while the component \( \bar{a}(\tau) \) lies in the area \( \mathbb{D} \) together with its \( p \)-neighbourhood.

Then for quite small \( \epsilon_0 > 0 \) a unique solution for the problem (2)-(6) exists, and for all \( x \in \mathbb{R}, t \in [0, L], \epsilon \in [0, \epsilon_0] \) the estimation is correct for deviation of solutions for the ordinary and averaged problems

\[ |u(\tau, \epsilon) - \bar{u}(\tau)| + |\varphi(\tau, \epsilon) - \bar{\varphi}(\tau, \epsilon)| + |\bar{u}(x, \tau, \epsilon) - \bar{u}(x, \tau)| \leq c \epsilon^a, \]

\[ |\epsilon(\tau)| \leq c \epsilon^a, \quad a = (mq)^{-1}. \]

**Note.** If Wronskian determinant has isolated zeros on \([0, L] \}, multiplicity of which does not outnumber \( k \), then by analogy as in [9], it is proved that the estimation has \((mq + k)^{-1}\) order.

**Example.** Let us consider the example of the two-frequency problem

\[ \begin{align*}
\frac{d^2u}{dt^2} & = \frac{d^2u}{dt^2} + \cos(k\varphi_1 + l\varphi_2), \\
\frac{d\varphi_1}{dt} & = \frac{d\varphi_1}{dt} + 2t \frac{d\varphi_2}{dt} = 1 + \tau, \\
\varphi_1(0) & = 0, \quad \frac{d\varphi_2}{dt} = \frac{-3}{4} \left( 45 + \frac{334}{e} \right), \\
\frac{1}{4} \int_0^1 \varphi_1(\tau)dt + \frac{1}{3} \int_0^1 \varphi_2(\tau)dt = d_\pi = \frac{1}{2}.
\end{align*} \]

If \( 2k = -l = 2, \theta = 0.5 \), then the resonance exists in the system when \( \tau = 0, \) as \( y_{12}(\tau) = 0.5\tau. \) Here

\[ u(x, 1, \epsilon) - \bar{u}(x, 1) = 2 \frac{\epsilon}{\sqrt{3}} \int_0^1 \cos x^2 dz = O(\sqrt{\epsilon}), \]

based on estimation of Fresnel integral.

If \( 8k = -l = 8, \theta = 0.5 \), then \( y_{12}(\tau) \) is resonance is absent in the system. Then

\[ u(x, 1, \epsilon) - \bar{u}(x, 1) = \frac{4\epsilon}{3} \sin \frac{3}{4\epsilon} = O(\epsilon). \]

**References**


