About Averaging in Hyperbolic Equation under the Influence of Multifrequence Disturbances

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Abstract— **The research deals with the existence of the solution of the initial problem for hyperbolic equation under the multifrequency disturbances, which are described by the system of ordinary differential equations (ODE) with multipoint and integral conditions. The averaging method over fast variables is grounded and estimation of accuracy of the method which obviously depends on the small parameter was found.**

Keywords—averaging method; multifrequency problem; system; resonance; small parameter; integral conditions.

Mathematical models in the problems of managing dynamic systems, in particular with delay, under the influence of multifrequency disturbances, string generators, and processes in chemical kinetics are described with the help of differential equations with ordinary and partial derivatives [1–3].

In this paper we consider the linear hyperbolic equation with delay

$$u_{tt} = c^2 u_{zz} + \mu \sum_{\nu=1}^{s} b_\nu (x,\tau) u_{\delta_\nu} + \mu f(x,\tau,a_\Lambda,\varphi_\Theta)$$

under the influence of multi-frequency disturbances, which are determined by the system of equations

$$\frac{da}{d\tau} = X(\tau,a_\Lambda,\varphi_\Theta),$$
$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y(\tau,a_\Lambda,\varphi_\Theta).$$

Here $\tau = \varepsilon t \in [0,L], x = \varepsilon z \in \mathbb{R}, \varepsilon \in (0,\varepsilon_0], \varepsilon_0 \ll 1, \mu = \varepsilon^2$, $a_\Lambda(\tau) = \left(a(\lambda_1\tau),\dots,a(\lambda_p\tau)\right), \varphi_\Theta(\tau) = \left(\varphi(\theta_1\tau),\dots,\varphi(\theta_q\tau)\right), u_\Delta(x,\tau) = \left(u(x,\delta_1\tau),\dots,u(x,\delta_r\tau)\right)$, $\lambda_i,\theta_j,\delta_k \in (0,1)$.

Moving to the slow variables in the equation (1) we will get the equation

$$u_{\tau\tau} = c^2 u_{\tau\tau} + \sum_{\nu=1}^{s} b_\nu (x,\tau) u_{\delta_\nu} + f(x,\tau,a_\Lambda,\varphi_\Theta).$$

For the solutions of the system of equations the following conditions are given:

$$u(x,0) = \varphi(x), u_\tau(x,0) = \psi(x), x\epsilon\mathbb{R}$$

$$\sum_{\nu=1}^{r} a(t_\nu)\, a(t_\nu) = d_1, \qquad 0 \le t_1 < \cdots < t_r \le L,$$

$$\int_0^{\tau_1} \left[\sum_{\nu=1}^{q} g_{1\nu}\big(\tau,a_\Lambda(\tau)\big)\varphi_{\theta_\nu}(\tau) + f_1\big(\tau,a_\Lambda(\tau),\varphi_\Theta(\tau)\big)\right] d\tau +$$

$$+ \int_{\tau_2}^{L} \left[\sum_{\nu=1}^{q} g_{2\nu}\big(\tau,a_\Lambda(\tau)\big)\varphi_{\theta_\nu}(\tau) + f_2\big(\tau,a_\Lambda(\tau),\varphi_\Theta(\tau)\big)\right] d\tau =$$

$$= d_2.$$

Averaging in the problem (3)–(6) is realized over fast variables on the cube of periods for equations (2), (3), as well as for integral conditions (5), (6). The averaging system takes the form of

$$\bar{u}_{\tau\tau} = c^2 \bar{u}_{xx} + \sum_{\nu=1}^{s} b_\nu (x,\tau) \bar{u}_{\delta_\nu} + f_0(x,\tau,\bar{a}_\Lambda),$$
$$\frac{d\bar{a}}{d\tau} = X_0(\tau,\bar{a}_\Lambda),$$
$$\frac{d\bar{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y_0(\tau,\bar{a}_\Lambda).$$

In boundary conditions (6) vector-functions f_1 and f_2 are averaged over the fast variables. As a result, a much simpler problem is received, because the equation for parabolic variables $\bar{a}$ does not depend on $\bar{\varphi}_\Theta$, finding $\bar{\varphi}$ is reduced to the problem of integration.

The main condition of the research of the multifrequency systems is the condition of passage of the system through resonance. For the system (1) the condition of resonance $\left(k, \omega\big(\tau, a(\tau,\varepsilon)\big)\right) \approx 0, \mathbb{Z}^m \ni k \neq 0, (\cdot,\cdot)$ – is a scalar product. For the system with transformed arguments a condition is found [4]:

$$\sum_{\nu=1}^{q} \theta_\nu(k_\nu, \omega(\theta_\nu\tau)) = 0, k_\nu \in \mathbb{R}^m, \|k\| \neq 0.$$

If $V(\tau) \neq 0, \tau \in [0,L]$, where $V(\tau)$ – is Wronsky determinant by the system of functions $\{\omega(\theta_1\tau),\dots,\omega(\theta_q\tau)\}$, then the system gets out of resonance.

For one-frequency system with one linearly transformed argument the Wronsky determinant is

$$V(\tau)=\begin{vmatrix}\omega(\tau) & \omega(z)\\ \dfrac{d\omega(\tau)}{d\tau} & \theta\dfrac{d\omega(z)}{dz}\end{vmatrix}, z=\theta\tau, \theta\in(0,1).$$

The grounding of the averaging method is grounded on the estimation of the corresponding to the system (2) oscillation integral.

For the ordinary differential equations such an estimation is received in [5], for the systems with linearly transformed arguments used in [4, 6, 7]. The grounding of existence and uniqueness of the solution of the problem (2), (5), (6) is founded on the basis of sufficient conditions for existence of the solution of the initial problem. Herewith we receive the estimation of accuracy of the method, the order of which is $\varepsilon^{\alpha}, \alpha=(mq)^{-1}$.

To prove the main statement we use the following integral inequality, which is synthesis of the result [8].

Lemma. Let the constant $\alpha\geq 0$, functions $\alpha_\nu\in C^1(I,I)$, $\alpha_\nu(t)\leq t$ for $t\in I$, $a_\nu\in C(I,R_+)$, $\nu=\overline{1,n}$.

If $u\in C(I,R_+)$ and

$$u(t)\leq\alpha+\sum_{\nu=1}^{q}\int_{\alpha_\nu(t_0)}^{\alpha_\nu(t)}a_\nu(s)u(s)ds,$$

then for $t\in I$

$$u(t)\leq\alpha\cdot\exp\left(\sum_{\nu=1}^{q}\int_{\alpha_\nu(t_0)}^{\alpha_\nu(t)}a_\nu(s)ds\right).$$

Corollary. If $t_0=0, \alpha_\nu(t)=\lambda_\nu t, \lambda_\nu\epsilon(0,1), \alpha_\nu\geq 0$, then

$$u(t)\leq\alpha\cdot\exp\left(\sum_{\nu=1}^{q}\alpha_\nu\lambda_\nu t\right).$$

Theorem. Suppose, that conditions are true:

1) vector-functions $X, Y, f_i, g_{i\nu}, i=1,2, \nu=\overline{1,q}$ and function f are differentiable over the variables x,τ,a_Λ and mq+1 once differentiable over the fast variables φ_θ;

2) $\omega_\nu\in C^{mq+1}[0,L], \nu=\overline{1,m}$, and Wronsky determinant by the system of functions $\{\omega(\theta_1\tau),\dots,\omega(\theta_m\tau)\}$ is not equal to zero on [0, L];

3) $\varphi\in C^2(\mathbb{R}), \psi\in C^1(\mathbb{R})$;

4) functions $b_\nu\in C(\mathbb{R}\times(0,1)), \nu=\overline{1,s}$;

5) a unique solution exists for the averaged problem, while the component $\bar{a}(\tau)$ lies in the area $\mathbb{D}$ together with its ρ–neighbourhood.

Then for quite small $\varepsilon_0>0$ a unique solution for the problem (2)-(6) exists, and for all $x\in\mathbb{R}, \tau\in[0,L], \varepsilon\in[0,\varepsilon_0]$ the estimation is correct for deviation of solutions for the ordinary and averaged problems

$$\|a(\tau,\varepsilon)-\bar{a}(\tau)\|+\|\varphi(\tau,\varepsilon)-\bar{\varphi}(\tau,\varepsilon)-\xi(\varepsilon)\|+$$
$$+|u(x,\tau,\varepsilon)-\bar{u}(x,\tau)|\leq\sigma\varepsilon^{\alpha},$$
$$\|\xi(\varepsilon)\|\leq c\varepsilon^{\alpha-1},\qquad \alpha=(mq)^{-1}.$$

Note. If Wronsky determinant has isolated zeros on [0, L], multiplicity of which does not outnumber κ, then by analogy as in [9], it is proved that the estimation has $(mq+\kappa)^{-1}$ order.

Example. Let us consider the example of the two-frequency problem

$$\frac{\partial^2 u}{\partial\tau^2}=\frac{\partial^2 u}{\partial\tau^2}+\cos(k\varphi_1+l\varphi_2),$$
$$\frac{\partial\varphi_1}{\partial\tau}=\frac{1+2\tau}{\varepsilon}, \frac{\partial\varphi_2}{\partial\tau}=\frac{1+\tau}{\varepsilon},$$
$$\varphi_1(0)=d_1=-4\left(45+\frac{334}{\varepsilon}\right),$$
$$\int_0^{1/4}\varphi_1(\tau)d\tau+\int_{3/4}^{1}\varphi_2(\tau)d\tau=d_2=\frac{1}{2}$$

If $2k=-l=2, \theta=0.5$, then the resonance exists in the system when τ = 0, as $\gamma_{12}(\tau)=0.5\tau$. Here

$$u(x,1,\varepsilon)-\bar{u}(x,1)=2\sqrt{\frac{\varepsilon}{3}}\int_0^{\frac{\sqrt{3}}{2\sqrt{z}}}\cos z^2\,dz=O(\sqrt{\varepsilon}),$$

based on estimation of Fresnel integral.

If $8k=-l=8, \theta=0.5$, then $\gamma_{(1,-1)}(\tau)=-3$ and the resonance is absent in the system. Then

$$u(x,1,\varepsilon)-\bar{u}(x,1)=\frac{4\varepsilon}{3}\sin\frac{3}{4\varepsilon}=O(\varepsilon).$$

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