Modeling Vector Fields in Space of Affine Connection

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Abstract— **The authors have been constructed the splitting of the basic geometric images vector field (points, straights, hyperplanes and hyperguadrics) in transition from n-dimensional affine space to the space of affine connection. All invectigations have been fulfilled in the moving coordinate system of zero order.**

Keywords— vector field, splitting of geometric image, affine space, hyperplane, hyperguadric.

I. Introduction

Problem formation. The research of manifolds of homogenious and generalized spaces is connected with definition of invariant geometric images of affine connection (straights, points, k-dimensional planes, hyperquadrics, etc). It is also related to the research of vector fields in n-dimensional space of affine connection.

Publication review on the subject. The investigation of vector fields and associated with them chapters in equiaffine space is connected with the activities of a number of scientists. We should mention Yurii A. Aminov [1], D.A. Sintsov [2], S.S.Bushgues [3] and many others. In case of three-dimensional affine space this problem was investigated by V.V.Slukhayev [4] and Ch. Gheorgiev [5]. The bases of differential geometry of vector field in n-dimensional affine space have been built in [6], in this case accompanying moving coordinate system is chosen by means of the case when one of the vectors coincides with vector of vector field.

Besides the previous information we remark that the last mentioned work deals with building of invariant linear models: points, straights, hyperplanes and hyperquadrics.

II. Formation of the aims of research:

To build invariant models for vector field in n-dimensional affine space with help of G.F.Laptiev [7] method.

The objective of the thesis is to receive the splitting of basic geometric images, vector fields in transition from n-dimensional affine space to n-dimensional space of affine connection.

III. Materials and methods

Classical space of affine connection $A_{n,n}$ is determined by system forms ω^i and ω^i_j satisfy the structure equations:

$$\begin{aligned} D\omega^\alpha &= \left[\omega^\beta \omega^\alpha_\beta\right] + R^\alpha_{\beta\gamma}\left[\omega^\beta \omega^\gamma\right], \\ D\omega^\alpha_\beta &= \left[\omega^\gamma_\beta \omega^\alpha_\gamma\right] + R^\alpha_{\beta\gamma\delta}\left[\omega^\gamma \omega^\delta\right], \left(\alpha,\ \beta,\ \gamma,\ \delta = \overline{1,n}\right) \end{aligned} \qquad (1.1)$$

In equations (1.1) values $R^\alpha_{\beta\gamma}$ are skew symmetric on subscript indexes and in a set they form torsion tensor of $A_{n,n}$ space, values $R^\alpha_{\beta\gamma\delta}$ are skew symmetric on indexes γ, δ and they form tensor of curvature of space $A_{n,n}$.

The space of affine connection A_n is the fibering space which base is considered to be any differential manifold M_n, and its layers are some central affine spaces $A_n(u)$ related to affine moving coordinate system $\{A_0(u), \vec{e}_\alpha, (u)\}$.

In this case n independent first integrals $u^1, u^2, ..., u^n$ of fully integrated system of forms ω^α are local coordinates of point $A(u^\alpha(u))$ of base M_n space $A_{n,n}$.

The forms ω^α, ω^α_β invariantly denote infinitively close affine reflection of neibouring local space (layer) into the given one with the help of moving coordinate system reflection

$$\begin{aligned} &A(u+du) \to dA(u) = \omega^\alpha \vec{e}_\alpha(u), \\ &e_\alpha(u+du) = e_\alpha(u) + de_\alpha(u) = \vec{e}_\alpha(u) + \omega^\beta_\alpha \vec{e}_\beta(u) \end{aligned} \qquad (1.2)$$

Definition. Vector field in space $A_{n,n}$ is called the function in which every point $A(u)$ base of space $A_{n,n}$ corresponds definite vector $\vec{v}(u)$ which belongs to n-dimensional affine space $A_n(u)$ related to moving coordinate system $T_n = \{\vec{A}(u), \vec{e}_\alpha(u)\}$. This space, as it is obvious, is a layer over point $A(u)$.

System of differential equations of vector field in moving coordinate system of zero order (starting point $A(u)$ of vector field coincides with the end of vector $\vec{A}$, and vector $\vec{v}$ coincides with l_n have the form

$$\alpha^\alpha_n = \Lambda^\alpha_{n\beta}\omega^\beta \ (\alpha, \beta, \gamma, ... = \overline{1,n}). \qquad (1.3)$$

Continuing the system of differential equations (1.3) we'll receive the system of differential equations of fundamental object of the first order of vector field of space $A_{n,n}$ in the form

$$d\Lambda^\alpha_{n\beta} = \Lambda^\alpha_{n\gamma}\omega^\gamma_\beta - \Lambda^\gamma_{n\beta}\omega^\alpha_\gamma + \Lambda^\alpha_{n\beta\beta}\omega^\gamma, \qquad (1.4)$$

where

$$\Lambda^{\alpha}_{n[\beta\gamma]} + \Lambda^{\alpha}_{n\delta} R^{\delta}_{\gamma\beta} + R^{\alpha}_{n\beta\beta} = 0 . \tag{1.5}$$

Continuing the system of differential equations (1.3) we'll have

$$d\Lambda^{\alpha}_{n\beta\beta} = \Lambda^{\alpha}_{n\delta\delta}\omega^{\delta}_{\beta} + \Lambda^{\gamma}_{n\beta\beta}\omega^{\alpha}_{\gamma} - \Lambda^{\delta}_{n\beta\beta}\omega^{\alpha}_{\delta} + \Lambda^{\alpha}_{n\beta\beta\gamma}\omega^{\delta} . \tag{1.6}$$

Succession of fundamental objects $\{\Lambda^{\alpha}_{n\beta}, \Lambda^{\alpha}_{n\beta\beta}, \Lambda^{\alpha}_{n\beta\beta\gamma}, ...\}$ lies on the basis of differential geometry of vector field in space $A_{n,n}$.

Remark. Apart from n-dimensional affine space tensors $\Lambda^{\alpha}_{n\beta\beta}, \Lambda^{\alpha}_{n\beta\beta\gamma}, ...$ loose symmetric properties on two down subscript indexes.

IV. Results and discussion

Find differential equations of some invariant geometric objects joint to vector field.

Field of points. Consider point $P(x^{\alpha})$ in affine space A_n . If $\vec{P}$ – is radius-vector of this point, in this case related to affine moving coordinate system $(\vec{A}, \overrightarrow{e_{\alpha}})$ it can be expressed with the following correspondence

$$\vec{P} = \vec{A} + x^{\alpha}\vec{e}_{\alpha} . \tag{2.1}$$

After differentiation of (2.1) taking into account equation of structure, receive

$$dx^{\alpha} + x^{\beta}\omega^{\alpha}_{\beta} = x^{\alpha}_{\beta}\omega^{\beta} , \tag{2.2}$$

or in fixation of main parameters

$$\delta x^{\alpha} + x^{\beta}\pi^{\alpha}_{\beta} = 0 . \tag{2.3}$$

Consider values

$$N^{\alpha} = \Lambda^{\alpha}_{nn} . \tag{2.4}$$

If differential equations values (2.4) have the form $dN^{\alpha} + N^{\beta}\omega^{\alpha}_{\beta} = N^{\alpha}_{\beta}\omega^{\beta}$ according to (2.2) they define invariant point.

Field of straights. Straight, which crosses the point A with directed vector $\vec{R} = v^{\alpha}\vec{e}_{\alpha}$ define as $l = [A, \vec{R}]$.

Conditions of invariance of straight will be

$$\delta\vec{R} = Q\vec{R}, dQ = 0 . \tag{2.5}$$

From the previous data

$$\delta v^{\alpha} + v^{\beta}\pi^{\alpha}_{\beta} = Qv^{\alpha} , \tag{2.6}$$

or

$$\begin{aligned} &\delta v^{i} + v^{i}\pi^{i}_{j} = Qv^{i}, \\ &\delta v^{n} + v^{i}\pi^{n}_{i} = Qv^{n}, (i = \overline{1, n-1}) \end{aligned} . \tag{2.7}$$

Sometimes it is convenient to promote norm of vector $\vec{R}$ in which $v^{n} = 1$.

Then

$$Q = v^{i}\omega^{n}_{i} . \tag{2.8}$$

Putting (2.8) in the first equation (2.7) we have

$$\delta v^{i} + v^{i}\pi^{i}_{j} - v^{i}v^{k}\pi^{n}_{k} = 0 . \tag{2.9}$$

Thus, differential equations of invariance of straight have the form

$$dv^{i} + v^{i}\omega^{i}_{j} - v^{i}v^{k}\omega^{n}_{k} = v^{i}_{\alpha}\omega^{\alpha} . \tag{2.10}$$

Build values $V^{\beta}_{n\alpha}$ (in condition of $def//\alpha^{\alpha}_{\beta}// \neq 0$)

$$\Lambda^{\gamma}_{n\alpha}V^{\beta}_{n\gamma} = \delta^{\beta}_{\alpha} , \tag{2.11}$$

and with their help values

$$\begin{aligned} &M^{\alpha} = -V^{\alpha}_{n\beta}\Lambda^{\beta}_{nn}, \\ &dM^{\alpha} = -M^{\beta}\omega^{\alpha}_{\beta} + M^{\alpha}_{\gamma}\omega^{\gamma} \end{aligned} . \tag{2.12}$$

If differential equations of (2.12) we have the equations' structure (2.6), the pair $[A, \vec{M}]$, where $\vec{M} = M^{\alpha}\vec{e}_{\alpha}$ defines invariant straight.

Statement 1. In differential neighborhood of the first order exists straight invariant with vector field which is determined by tensor M^{α} .

Field of hyperplanes. Equation of condition of invariance of hyperplane $v_{\alpha}x^{\alpha} + v = 0$ related to moving coordinate system $(\vec{A}, \vec{e}_{\alpha})$ has the following form

$$\begin{aligned} &\delta v_{\alpha} + v_{\beta}\pi^{\beta}_{\alpha} = Qv_{\alpha}, \\ &\delta v = Qv. \end{aligned} \tag{2.13}$$

Q − a linear form, which means $dQ = 0$. Two cases are possible.

1. Hyperplane doesn't cross point A.

It's possible to put in this case $v = 1$, then $Q = 0$. Conditions of invariance of hyperplane take the form

$$\delta v_{\alpha} + v_{\beta}\pi^{\beta}_{\alpha} = 0 . \tag{2.14}$$

2. Hyperplane crosses point A.

In this case $v = 0$. Conditions of its invariance have the form

$$\delta v_{\alpha} - v_{\beta}\pi^{\beta}_{\alpha} = Qv_{\alpha} . \tag{2.15}$$

Putting $v_n = 1$ conditions (2.15) have the form

$$\delta v_{i} - v_{j}\pi^{j}_{i} - \pi^{n}_{i} = 0 . \tag{2.16}$$

Build values $g^{\alpha\beta} = N^{\alpha}M^{\beta}$.

In condition of $def//g^{\alpha\beta}// \neq 0$ introduce values $g^{\alpha\delta}g_{\gamma\beta} = \delta^{\alpha}_{\beta}$ and with their help

$$\begin{aligned} &g_{\alpha} = g_{\alpha\beta}N^{\alpha}, \\ &\delta g_{\alpha} - g_{\beta}\pi^{\beta} = 0. \end{aligned} \tag{2.17}$$

If differential equations of value g_{α} have structure of differential equations (2.14) then these values define hyperplane which doesn't cross point A in the form $g_{\alpha}x^{\alpha} = 0$.

Statement 2. In differential neighborhood of the first order exists invariant hiperplane of vector field which doesn't cross its beginning and which is determined by tensor g_{α} .

We build invariant hyperplanes with help of fundamental tensor of the second order Λ^{α}_{nji}. With this

$$\begin{aligned} \overset{1}{\Lambda}_{n\alpha} &= \Lambda^{\beta}_{n\alpha\alpha}, \\ \delta \overset{1}{\Lambda}_{n\alpha} - \overset{1}{\Lambda}_{n\beta}\, \pi^{\beta}_{\alpha} &= 0 \end{aligned} \qquad (2.17)$$

and also tensor

$$\begin{aligned} \overset{2}{\Lambda}_{n\alpha} &= \Lambda^{\beta}_{n\beta\beta}, \\ \delta \overset{2}{\Lambda}_{n\alpha} - \overset{2}{\Lambda}_{n\beta}\, \pi^{\beta}_{\alpha} &= 0 \end{aligned} \qquad (2.19)$$

Obviously, tensors (2.18) and (2.19) for space $A_{n,n}$ are different, and in case of affine space A_n coincide. They denote invariant hyperplanes that do not cross the beginning of forming element and they are also defined by the equations:

$$\overset{1}{\Lambda}_{n\alpha}\, x^{\alpha} + 1 = 0\,, \qquad (2.20)$$

$$\overset{2}{\Lambda}_{n\alpha}\, x^{\alpha} + 1 = 0 \qquad (2.21)$$

Hyperplanes (2.20) and (2.21) are called the main hyperplanes of vector field.

Statement 3. In case of transition from vector field of n-dimensional space A_n to vector field in *n*-dimensional space of affine connection the splitting of main hyperplanes takes place.

Differential conditions of invariance of hyperquadric joint to vector field.

The equation of hyperquadric relatively to any local moving coordinate system $\left(\vec{A}, \vec{e}_{\alpha}\right)$ has the form

$$A_{\alpha\beta} x^{\alpha} x^{\beta} + 2A_{\alpha} x^{\alpha} + A = 0\,. \qquad (3.1)$$

Where $A_{\alpha\beta} = A_{\beta\alpha}$.

Hyperquadric doesn't cross the beginning of definite vector element.

In this case it is possible to take $A = 1$. Thus, $\Theta = 0$.

Equations of invariance of hyperquadric obtain the form

$$\begin{aligned} &\delta A_{\alpha\beta} - A_{\alpha\gamma}\pi^{\gamma}_{\beta} - A_{\gamma\beta}\pi^{\gamma}_{\alpha} = 0, \\ &\delta A_{\alpha} - A_{\gamma}\pi^{\lambda}_{\alpha} = 0. \end{aligned} \qquad (3.2)$$

Hyperquadrics cross the beginning of forming vector element.

In this case $A = 0$ and conditions of invariance of quadric have the form

$$\begin{aligned} &\delta A_{\alpha\beta} - A_{\alpha\gamma}\pi^{\gamma}_{\beta} - A_{\gamma\beta}\pi^{\gamma}_{\alpha} = \Theta A_{\alpha\beta}, \\ &\delta A_{\alpha} - A_{\gamma}\pi^{\lambda}_{\alpha} = \Theta A_{\alpha} \end{aligned} . \qquad (3.3)$$

Considering the general form we put $A_n = 1$. Then $\Theta = 0$. Conditions of invariance of hyperquadric that crosses the beginning of vector element will be:

$$\begin{aligned} &\delta A_{ij} - A_{ik}\pi^{k}_{j} - A_{kj}\pi^{k}_{i} - A_{in}\pi^{n}_{j} - A_{nj}\pi^{n}_{i} = 0, \\ &\delta A_{in} - A_{kn}\pi^{k}_{i} - A_{nn}\pi^{n}_{i} = 0, \\ &\delta A_{nn} = 0, \\ &\delta A_{i} - A_{k}\pi^{k}_{i} - \pi^{n}_{i} = 0. \end{aligned} \qquad (3.4)$$

The hyperquadric proper is taken by the equation:

$$\begin{aligned} &A_{ij} x^i x^j + 2A_{in} x^i x^n + \\ &+ A_{nn}(x^n)^2 + 2A_i x^i + 2x^n = 0. \end{aligned} \qquad (3.5)$$

Some types of invariant hyperquadrics joint to vector field.

Invariant hyperquadric which is defined by the fundamental objects of the first and second order.

4. Analyze tensor $\Lambda^{\alpha}_{n\beta}$. In general case.

$$\Lambda = det\left|\Lambda^{\alpha}_{n\beta}\right| \neq 0\,. \qquad (4.1)$$

It permits to introduce values $v^{\alpha}_{n\beta}$, that their components are defined from the correspondence,

$$\Lambda^{\alpha}_{n\beta} v^{\alpha}_{n\beta} = \delta^{\alpha}_{\beta}\,. \qquad (4.2)$$

Differential equations of tensor have the form

$$dv^{\alpha}_{n\beta} = v^{\alpha}_{n\gamma}\omega^{\gamma}_{\beta} - v^{\gamma}_{n\beta}\omega^{\alpha}_{\gamma} + v^{\alpha}_{n\beta\beta}\omega^{\gamma}\,. \qquad (4.3)$$

Build tensor

$$\Gamma^{\alpha}_{\beta\gamma} = v^{\alpha}_{n\delta}\Lambda^{\delta}_{n\beta\beta}\,. \qquad (4.4)$$

Its differential equations in case of fixation of main parameters obtain the form

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}\pi^{\delta}_{\beta} + \Gamma^{\alpha}_{\beta\delta}\pi^{\delta}_{\gamma} - \Gamma^{\delta}_{\beta\gamma}\pi^{\alpha}_{\delta} \qquad (4.5)$$

due to tensor which is defined by the correspondence (4.4.) consequently build tensors

$$\begin{aligned} &\Gamma_{\alpha} = \Gamma^{\beta}_{\beta\alpha}, \\ &\delta\Gamma_{\alpha} - \Gamma_{\beta}\pi^{\beta}_{\alpha} = 0. \end{aligned} \qquad (4.6)$$

$$\begin{aligned} &\Gamma_{\alpha\beta} = \Gamma_{\gamma}\Gamma^{\gamma}_{\alpha\beta}, \\ &\delta\Gamma_{\alpha\beta} - \Gamma_{\gamma\beta}\pi^{\gamma}_{\alpha} - \Gamma_{\alpha\gamma}\pi^{\gamma}_{\beta} = 0. \end{aligned} \qquad (4.7)$$

If differential equations of tensors (4.6.) and (4.7.) have structure of equations (3.4.) the equations of invariant hyperquadric which is determined by the fundamental objects of the first and second order and it doesn't cross the beginning of definite vector element has the form

$$\Gamma_{\alpha\beta} x^{\alpha} + 2\Gamma_{\alpha} x^{\alpha} + 1 = 0\,. \qquad (4.8)$$

2. Invariant hyperquadric which is determined by the fundamental object of the second order.

Owing to tensor $\Lambda^{\alpha}_{n\beta\beta}$ build tensors

$$\begin{aligned} &\overset{1}{\Lambda}_{n\alpha\alpha} = \Lambda^{\gamma}_{n\alpha\alpha}\,\overset{1}{\Lambda}_{n\gamma}\,, \\ &\delta\overset{1}{\Lambda}_{n\alpha\alpha} = \overset{1}{\Lambda}_{n\gamma\gamma}\,\pi^{\gamma}_{\alpha} + \overset{1}{\Lambda}_{n\alpha\alpha}\pi^{\gamma}_{\beta}. \end{aligned} \qquad (4.9)$$

$$\begin{aligned} &\overset{2}{\Lambda}_{n\alpha\alpha} = \Lambda^{\gamma}_{n\alpha\alpha}\,\overset{2}{\Lambda}_{n\gamma}\,, \\ &\delta\overset{2}{\Lambda}_{n\alpha\alpha} = \overset{2}{\Lambda}_{n\alpha\alpha}\pi^{\gamma}_{\beta} + \overset{2}{\Lambda}_{n\gamma\gamma}\,\pi^{\gamma}_{\alpha}. \end{aligned} \qquad (4.10)$$

If differential equations of tensors (4.9.) and (4.10.) have structure of equations (3.4.) equation of invariant hyperquadrics which is determined by the fundamental object of the second order and it doesn't cross the beginning of definite vector element and it has the form

$$\overset{1}{\Lambda}_{n\alpha\alpha}\, x^{\alpha} x^{\beta} + 2\overset{1}{\Lambda}_{n\alpha}\, x^{\alpha} + 1 = 0\,. \qquad (4.11)$$

$$\overset{2}{\Lambda}_{n\alpha\alpha}\, x^{\alpha} x^{\beta} + 2\overset{2}{\Lambda}_{n\alpha}\, x^{\alpha} + 1 = 0\,. \qquad (4.12)$$

Definition. Quadric (4.11) and (4.12) is called the main hyperquadrics.

Statement 4. In case of transition from vector field of n-dimensional affine space A_n to vector field of n-dimensional space of affine connection the splitting of main hyperquadric takes place.

V. CONCLUSION

It has been constructed the splitting of the basic geometric images vector field (points, straights, hyperplanes and hyperguadrics) in transition from dimensional affine space to the space of affine connection. All investigations have been fulfilled in the moving coordinate system of zero order.

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